

# REPRESENTATIONS OF THE DOUBLE BURNSIDE ALGEBRA AND COHOMOLOGY OF THE EXTRASPECIAL $p$ -GROUP II

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**ABSTRACT.** Let  $E$  be the extraspecial  $p$ -group of order  $p^3$  and exponent  $p$  where  $p$  is an odd prime. We determine the mod  $p$  cohomology  $H^*(X, \mathbb{F}_p)$  of a summand  $X$  in the stable splitting of  $p$ -completed classifying space  $BE$ . In the previous paper [Representations of the double Burnside algebra and cohomology of the extraspecial  $p$ -group, J. Algebra 409 (2014) 265-319], we determined these cohomology modulo nilpotence. In this paper, we consider the whole part of the cohomology. Moreover, we consider the stable splittings of  $BG$  for some finite groups with Sylow  $p$ -subgroup  $E$  related with the three dimensional linear group  $L_3(p)$ .

## 1. INTRODUCTION

Let  $p$  be an odd prime and  $E = p_+^{1+2}$  the extraspecial  $p$ -group of order  $p^3$  and exponent  $p$ . In the previous paper [7], we determined the composition factor of  $H^*(E) = (\mathbb{F}_p \otimes H^*(E, \mathbb{Z}))/\sqrt{(0)}$  as a right  $A_p(E, E)$ -module, where  $A_p(E, E)$  is a double Burnside algebra of  $E$  over  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ . In this paper, we consider the whole part of the cohomology  $H^*(E, \mathbb{F}_p)$  and determine the composition factor of  $H^*(E, \mathbb{F}_p)$  as an  $A_p(E, E)$ -module.

The mod  $p$  cohomology ring  $H^*(E, \mathbb{F}_p)$  of  $E$  is completely known by [9], but the structure is very complicated. We shall study  $H^*(E, \mathbb{F}_p)$  through the integral cohomology ring  $H^*(E, \mathbb{Z})$  as in [13] and [14]. Let  $H^{even}(E, \mathbb{Z})$  (resp.  $H^{odd}(E, \mathbb{Z})$ ) be the even (resp. odd) degree part of  $H^*(E, \mathbb{Z})$ . Let  $N = \sqrt{(0)}$  in  $\mathbb{F}_p \otimes H^{even}(E, \mathbb{Z})$ . Then we have that  $\mathbb{F}_p \otimes H^{even}(E, \mathbb{Z})/N \cong H^*(E)$ . On the other hand, the Milnor operator  $Q_1$  induces an isomorphism  $H^{odd}(E, \mathbb{Z}) \cong (y_1v, y_2v)H^*(E)$  where  $(y_1v, y_2v)H^*(E)$  is the ideal of  $H^*(E)$  generated by  $y_1v, y_2v \in H^{2p+2}(E)$  (see the first part of section 2).

Let  $M = \oplus M^n$  and  $L = \oplus L^n$  be graded  $A_p(E, E)$ -modules such that  $M^n$  and  $L^n$  are finite dimensional for every  $n$ . We write as  $M \leftrightarrow L$  if  $M^n$  and  $L^n$  have same composition factors (with same multiplicity), that is,  $[M^n] = [L^n]$  in the Grothendieck group  $K_0(A_p(E, E))$ .

Using this notation, the structure of  $H^*(E, \mathbb{F}_p)$  can be stated as follows.

**Theorem 1.1.** (1) As  $A_p(E, E)$ -modules,

$$H^{even}(E, \mathbb{F}_p) \leftrightarrow H^*(E) \oplus N \oplus (y_1v, y_2v)H^*(E)[-2p].$$

(2) As  $A_p(E, E)$ -modules,

$$H^{odd}(E, \mathbb{F}_p) \leftrightarrow (y_1v, y_2v)H^*(E)[-2p+1] \oplus (N \oplus H^+(E))[-1].$$

Here, for a graded  $\mathbb{F}_p$ -subspace  $M$  of  $H^*(E)$ , we denote by  $M[i]$  the graded vector space with  $M[i]^n = M^{n-i}$ . Since the composition factors of  $N$  and  $(y_1v, y_2v)H^*(E)$  are determined in Proposition 3.1 and Theorem 3.5, we can get the composition factors of  $H^*(E, \mathbb{F}_p)$  completely.

The indecomposable summands in the complete stable splitting of the  $p$ -completed classifying space  $BE_p^\wedge$  correspond to primitive idempotents in  $A_p(E, E)$ . Moreover they corresponds to simple  $A_p(E, E)$ -modules. We simply write  $BE$  for  $BE_p^\wedge$ . Let  $X$  be a summand in  $BE$  which corresponds to a simple  $A_p(E, E)$ -module  $S$ . Then the multiplicity of  $X$  in  $BE$  is equal to the dimension of  $S$  as an  $\mathbb{F}_p$ -vector space since  $\mathbb{F}_p$  is a splitting field for  $A_p(E, E)$ . By results above, we can get the cohomology  $H^*(X, \mathbb{F}_p)$  (See Remark 3.6).

Let  $G$  be a finite group with Sylow  $p$ -subgroup  $E$ . Then the multiplicity of  $X$  in  $BG$  is equal to the dimension of  $S[G]$  where  $[G]$  is an element of  $A_p(E, E)$  corresponds to the  $(E, E)$ -biset  $G$ . See [1], [2], [11] for details.

In [15], the second author studied the splitting of  $BG$  for various finite groups  $G$  whose Sylow  $p$ -subgroup is  $E$  and  $p$ -local finite groups on  $E$ . In this paper, we consider the stable splitting for groups related with the linear group  $L_3(p)$  which were not treated in [15] in general. We use some simple  $A_p(E, E)$ -submodules of  $H^*(E)$  and determine the multiplicity of summands in  $BG$  for  $G = L_3(p)$ ,  $L_3(p) : 2$ ,  $L_3(p).3$ ,  $L_3(p).S_3$  (Theorem 4.17, 4.18, 4.19, 4.20).

Combining these results and results in [15], we have the complete information on the stable splitting of finite groups or  $p$ -local finite groups which have at least two  $\mathcal{F}$ -radical maximal elementary abelian  $p$ -subgroups in  $E$ , by the classification in [12], where  $\mathcal{F}$  is a fusion system of  $G$ .

In particular, for  $p = 7$ , we obtain a diagram which describes inclusions of some fusion systems and stable splitting (Theorem 4.23). This result supplements the results of [15, section 9], in which the splitting of sporadic simple groups are mainly studied.

In section 2, we review the main results of [7] which will be used in section 3. In section 3, we prove Theorem 1.1 and determine the structures of ideals  $N$  and  $(y_1v, y_2v)H^*(E)$ . In section 4, we consider  $H^*(G)$  and the stable splitting for finite group  $G$  which has a Sylow  $p$ -subgroup  $E$ . Finally, in section 5, we consider the case  $p = 3$  and state some remarks.

## 2. PRELIMINARY RESULTS ON $H^*(E)$

In this section, we quote some results from [7]. Let  $p$  be an odd prime. Let

$$E = \langle a, b, c \mid [a, b] = c, a^p = b^p = c^p = [a, c] = [b, c] = 1 \rangle$$

be the extraspecial  $p$ -group of order  $p^3$  and exponent  $p$ . Let  $A_i = \langle c, ab^i \rangle$  for  $0 \leq i \leq p-1$  and  $A_\infty = \langle c, b \rangle$ . Then

$$\mathcal{A}(E) = \{A_0, A_1, \dots, A_{p-1}, A_\infty\}$$

is the set of all maximal elementary abelian  $p$ -subgroups of  $E$ .

The cohomology of  $E$  is known by [8], [9], [15]. In particular,  $H^*(E) = (\mathbb{F}_p \otimes H^*(E, \mathbb{Z}))/\sqrt{(0)}$  is generated by

$$y_1, y_2, C, v$$

with

$$\deg y_i = 2, \deg C = 2p - 2, \deg v = 2p$$

subject to the following relations:

$$y_1^p y_2 - y_1 y_2^p = 0, Cy_i = y_i^p, C^2 = y_1^{2p-2} + y_2^{2p-2} - y_1^{p-1} y_2^{p-1}.$$

We set  $V = v^{p-1}$  and  $Y_i = y_i^{p-1}$ .

Let  $R$  be a subalgebra of  $H^*(E)$  and  $x_1, \dots, x_r$  elements of  $H^*(E)$ . We set

$$R\{x_1, \dots, x_r\} = \sum_{i=1}^r Rx_i$$

if  $x_1, \dots, x_r$  are linearly independent over  $R$ . Moreover, if  $W = \sum_{i=1}^r \mathbb{F}_p x_i$  is a  $\mathbb{F}_p$ -vector space spanned by  $x_1, \dots, x_r$ , then we set

$$R\{W\} = R\{x_1, \dots, x_r\}.$$

We consider the action of  $\text{Out}(E) = \text{GL}_2(\mathbb{F}_p)$  on  $H^*(E)$ . Let  $S^i$  be the homogeneous part of degree  $2i$  in  $\mathbb{F}_p[y_1, y_2]$ . Then  $p(p-1)$  simple  $\mathbb{F}_p \text{Out}(E)$ -modules

$$S^i v^q \cong S^i \otimes (\det)^q \quad (0 \leq i \leq p-1, 0 \leq q \leq p-2)$$

give the complete set of representatives of nonisomorphic simple  $\mathbb{F}_p \text{Out}(E)$ -modules. Let us write

$$\mathbb{C}\mathbb{A} = \mathbb{F}_p[C, V]$$

and

$$\mathbb{D}\mathbb{A} = \mathbb{F}_p[D_1, D_2]$$

where  $D_1 = C^p + V$ ,  $D_2 = CV$ . Then  $\mathbb{C}\mathbb{A} = H^*(E)^{\text{Out}(E)}$ , the  $\text{Out}(E)$ -invariants, and the restriction map induces an isomorphism

$$\mathbb{D}\mathbb{A} \xrightarrow{\sim} H^*(A)^{\text{Out}(A)}$$

for all  $A \in \mathcal{A}(E)$ .

Let

$$T^i = \mathbb{F}_p\{y_1^{p-1}y_2^i, y_1^{p-2}y_2^{i+1}, \dots, y_1^i y_2^{p-1}\}$$

for  $1 \leq i \leq p-2$ . Then  $S^{p-1+i} = CS^i + T^i$ . The  $\mathbb{F}_p$ -subspace  $CS^i$  is a  $\text{GL}_2(\mathbb{F}_p)$ -submodule of  $CS^i + T^i$  and

$$(CS^i + T^i)/CS^i \cong (S^{p-1-i} \otimes \det^i).$$

Moreover we have the following expression [7, Theorem 4.4]:

$$H^*(E) = \mathbb{F}_p[C, v] \left\{ \left( \bigoplus_{i=0}^{p-1} S^i \right) \oplus \left( \bigoplus_{i=1}^{p-2} T^i \right) \right\} = \mathbb{C}\mathbb{A} \left\{ \bigoplus_{i=0}^{p-2} \bigoplus_{q=0}^{p-2} (S^i v^q \oplus T^i v^q) \right\}$$

where  $S^0 = \mathbb{F}_p$  and  $T^0 = S^{p-1}$ .

Let  $C_p$  be a cyclic group of order  $p$  and let  $U_i = H^{2i}(C_p, \mathbb{F}_p)$  ( $0 \leq i \leq p-2$ ). Then  $U_i$  are simple  $\mathbb{F}_p \text{Out}(C_p)$ -modules. Let  $A \in \mathcal{A}(E)$  be a maximal elementary abelian  $p$ -subgroup of  $E$ . Let  $S(A)^i = H^{2i}(A)$ . Then  $S(A)^i \otimes \det^q$  ( $0 \leq i \leq p-1, 0 \leq q \leq p-2$ ) are simple modules for  $\text{Out}(A) = \text{GL}_2(\mathbb{F}_p)$ .

Let  $P$  be a general finite  $p$ -group and  $A_p(P, P)$  the double Burnside algebra of  $P$  over  $\mathbb{F}_p$ . The simple  $A_p(P, P)$ -modules corresponds to some pairs  $(Q, V)$  where  $Q \leq P$  and  $V$  is a simple  $\mathbb{F}_p \text{Out}(Q)$ -module, see [2], [4], [11]. In this paper, we denote the simple  $A_p(P, P)$ -module corresponds to the pair  $(Q, V)$  by  $S(P, Q, V)$ .

On the other hand, Dietz and Priddy [5] studied the stable splitting of  $BE$  and determined the multiplicity of each summand. In particular, their result implies the classification of simple  $A_p(E, E)$ -modules.

**Proposition 2.1** ([5], [7, Proposition 10.1]). *The simple  $A_p(E, E)$ -modules are given as follows:*

(1)  $S(E, E, S^i \otimes \det^q)$  for  $0 \leq i \leq p-1$ ,  $0 \leq q \leq p-2$ ,

$$\dim S(E, E, S^i \otimes \det^q) = i + 1.$$

(2)  $S(E, A, S(A)^{p-1} \otimes \det^q)$  for  $0 \leq q \leq p-2$ ,

$$\dim S(E, A, S(A)^{p-1} \otimes \det^q) = p + 1.$$

(3)  $S(E, C_p, U_i)$  for  $0 \leq i \leq p-2$ ,

$$\dim S(E, C_p, U_i) = \begin{cases} p+1 & (i=0) \\ i+1 & (1 \leq i \leq p-2). \end{cases}$$

(4)  $S(E, 1, \mathbb{F}_p)$ ,  $\dim S(E, 1, \mathbb{F}_p) = 1$ .

To describe the composition factor of  $H^*(E)$  as an  $A_p(E, E)$ -module, we need the following  $\mathbb{F}_p$ -subspace of  $H^*(E)$ .

**Definition 2.2.** *Let  $S$  be a simple  $A_p(E, E)$ -module. Let  $\Gamma_S$  be the following  $\mathbb{F}_p$ -subspace of  $H^*(E)$ :*

(1) *If  $S = S(E, C_p, U_i)$ , then*

$$\Gamma_S = \begin{cases} \mathbb{F}_p[C]\{\mathbb{F}_p C + S^{p-1}\} & (i=0) \\ \mathbb{F}_p[C]\{S^i\} & (1 \leq i \leq p-2). \end{cases}$$

(2) *If  $S = S(E, A, S(A)^{p-1} \otimes \det^q)$ , then*

$$\Gamma_S = \begin{cases} \mathbb{D}\mathbb{A}\{\oplus_{0 \leq j \leq p-1} D_2 C^j (\mathbb{F}_p C + S^{p-1})\} & (q=0) \\ \mathbb{D}\mathbb{A}\{\oplus_{0 \leq j \leq p-1} v^q C^j (CS^q + T^q)\} & (1 \leq q \leq p-2). \end{cases}$$

(3)

$$\Gamma_S = \begin{cases} \mathbb{D}\mathbb{A}^+ & (S = S(E, E, S^0)) \\ \mathbb{C}\mathbb{A}\{v^q\} & (S = S(E, E, \det^q), 1 \leq q \leq p-2) \\ \mathbb{D}\mathbb{A}\{VS^{p-1}\} & (S = S(E, E, S^{p-1})) \\ \mathbb{C}\mathbb{A}\{v^q S^{p-1}\} & (S = S(E, E, S^{p-1} \otimes \det^q), 1 \leq q \leq p-2) \end{cases}$$

(4) *Let*

$$S = S^i v^q, \quad T = T^{p-i-1} v^s$$

*for  $1 \leq i \leq p-2$ ,  $0 \leq q \leq p-2$ , where  $s \equiv i+q \pmod{p-1}$ ,  $0 \leq s \leq p-2$ . Let  $\Gamma_{S(E, E, S^i \otimes \det^q)}$  be the following  $\mathbb{F}_p$ -subspace:*

$$\begin{aligned} \mathbb{C}\mathbb{A}\{VS\} &\oplus \mathbb{D}\mathbb{A}\{VT\} & (q \equiv 2i \equiv 0) \\ \mathbb{C}\mathbb{A}\{VS\} &\oplus \mathbb{C}\mathbb{A}\{T\} & (q \equiv 0, 2i \not\equiv 0) \\ \mathbb{D}\mathbb{A}\{S\} &\oplus \mathbb{D}\mathbb{A}\{VT\} & (i = q, 3i \equiv 0) \\ \mathbb{D}\mathbb{A}\{S\} &\oplus \mathbb{C}\mathbb{A}\{T\} & (i = q, 3i \not\equiv 0) \\ \mathbb{C}\mathbb{A}\{S\} &\oplus \mathbb{D}\mathbb{A}\{VT\} & (q \not\equiv 0, i \not\equiv q, q+2i \equiv 0) \\ \mathbb{C}\mathbb{A}\{S\} &\oplus \mathbb{C}\mathbb{A}\{T\} & (q \not\equiv 0, i \not\equiv q, q+2i \not\equiv 0). \end{aligned}$$

(5)

$$\Gamma_{S(E, 1, \mathbb{F}_p)} = \mathbb{F}_p = H^0(E).$$

The following theorem is the main result of [7]. If  $S$  is a simple  $A_p(E, E)$ -module, then there exists an idempotent  $e_S$  such that  $Se_S = S$  and  $S'e_S = 0$  for any simple module  $S' \not\cong S$ . We call  $e_S$  an idempotent which corresponds to  $S$ .

**Theorem 2.3** ([7, Theorem 10.2, 10.3, 10.4, 10.5]). *Let  $S$  be a simple  $A_p(E, E)$ -module. Then there exists an idempotent  $e_S$  which corresponds to  $S$  such that*

$$H^*(E)e_S = \Gamma_S e_S \cong \Gamma_S.$$

If we see the minimal degree of non zero part of  $\Gamma_S$ , we have the following corollary.

**Corollary 2.4.** *Every simple  $A_p(E, E)$ -module appears as a composition factor in  $H^{2n}(E)$  for some  $n \leq (p+2)(p-1)$ .*

Let  $\Gamma_S^n = \Gamma_S \cap H^n(E)$  be the degree  $n$  part of  $\Gamma_S$ . Then by Theorem 2.3,

$$\sum_S \dim \Gamma_S^n = \dim H^n(E)$$

for any  $n \geq 0$ . In fact we have the following.

**Proposition 2.5.**  *$H^*(E)$  is a direct sum of  $\mathbb{F}_p$ -subspaces  $\Gamma_S$  where  $S$  runs over the representatives of the isomorphism classes of simple  $A_p(E, E)$ -modules.*

*Proof.* By Theorem 2.3, it suffices to show that  $H^*(E) = \sum \Gamma_S$ . We shall show that  $\mathbb{CA}\{S^i v^q\}$  ( $0 \leq i \leq p-1$ ,  $0 \leq q \leq p-2$ ) and  $\mathbb{CA}\{T^i v^q\}$  ( $1 \leq i \leq p-2$ ,  $0 \leq q \leq p-2$ ) are contained in  $\sum \Gamma_S$ .

First consider  $\mathbb{CA}\{S^i v^q\}$ . If  $(i, q) = (0, 0)$ ,

$$\begin{aligned} \mathbb{CA} = \mathbb{F}[C, D_1] &= \mathbb{F}[D_1] \oplus \mathbb{CA}\{C\} \\ &= \mathbb{F}[D_1] \oplus \mathbb{F}[C]\{C\} \oplus \mathbb{CA}\{D_2\} \\ &= \mathbb{F}[D_1] \oplus \mathbb{F}[C]\{C\} \oplus \sum_{j=0}^{p-1} \mathbb{DA}\{D_2 C^{j+1}\} \oplus \mathbb{DA}\{D_2\} \\ &= \mathbb{DA} \oplus \mathbb{F}[C]\{C\} \oplus \sum_{j=0}^{p-1} \mathbb{DA}\{D_2 C^{j+1}\} \end{aligned}$$

and this is contained in the sum of the subspaces of Definition 2.2 (1)(2)(3)(5). If  $(i, q) = (0, q)$ ,  $1 \leq q \leq p-2$ , or  $(i, q) = (p-1, q)$ ,  $1 \leq q \leq p-2$ , then  $\mathbb{CA}v^q$  and  $\mathbb{CA}S^{p-1}v^q$  are contained in the subspace of Definition 2.2 (3). If  $(i, q) = (p-1, 0)$ , then

$$\begin{aligned} \mathbb{CA} &= \mathbb{F}[C] \oplus \mathbb{CA}\{V\} \\ &= \mathbb{F}[C] \oplus \mathbb{DA}\{V\} \oplus \mathbb{DA}\{D_2, D_2 C, \dots, D_2 C^{p-1}\} \end{aligned}$$

and we have

$$\mathbb{CA}\{S^{p-1}\} = \mathbb{F}_p[C]\{S^{p-1}\} \oplus \mathbb{DA}\{V S^{p-1}\} \oplus (\oplus_{j=0}^{p-1} \mathbb{DA}\{D_2 C^j S^{p-1}\}).$$

This is contained in the sum of subspaces of Definition 2.2 (1)(2)(3).

Consider  $(i, q)$  ( $1 \leq i \leq p-2$ ,  $0 \leq q \leq p-2$ ). If  $q = 0$ , then

$$\mathbb{CA}\{S^i\} = \mathbb{F}_p[C]\{S^i\} \oplus \mathbb{CA}\{V S^i\}$$

and this is contained in the sum of subspaces of Definition 2.2 (1)(4). If  $i = q$ , then

$$\mathbb{CA}\{S^i v^i\} = \mathbb{DA}\{S^i v^i\} \oplus (\oplus_{j=0}^{p-1} \mathbb{DA}\{C^{j+1} S^i v^i\})$$

and this is contained in the sum of the subspaces of Definition 2.2 (2)(4). If  $q \neq 0$  and  $i \neq q$ , then  $\mathbb{CA}\{S^i v^q\}$  is contained in the subspace of Definition 2.2 (4).

Next, consider  $T^k v^m$  ( $1 \leq k \leq p-2$ ,  $0 \leq m \leq p-2$ ). Let  $i = p-k-1$ ,  $q \equiv m+k \pmod{p-1}$ ,  $0 \leq q \leq p-2$ ,  $s = m$ . Then

$$T^k v^m = T^{p-i-1} v^s$$

where  $1 \leq i \leq p-2$ ,  $0 \leq q \leq p-2$ ,  $s \equiv i+q \pmod{p-1}$  and

$$q+2i \equiv m+k+2(p-k-1) \equiv m-k \pmod{p-1}.$$

If  $k \neq m$ , then  $q+2i \not\equiv 0$  and  $\mathbb{CA}\{T^k v^m\}$  is contained in the subspace of Definition 2.2 (4). If  $k = m$ , then  $q+2i \equiv 0$  and

$$\mathbb{CA}\{T^k v^m\} = \mathbb{DA}\{\oplus_{j=0}^{p-1} T^k C^j v^m\} \oplus \mathbb{DA}\{T^k V v^m\}$$

since  $\mathbb{CA} = \mathbb{DA}\{1, C, \dots, C^{p-1}, V\}$ . This is contained in the sum of the subspaces of Definition 2.2 (2)(4).  $\square$

### 3. COMPOSITION FACTORS OF $H^*(E, \mathbb{F}_p)$

In this section, we study the  $A_p(E, E)$ -module structure of  $H^*(E, \mathbb{F}_p)$ . First, we shall prove Theorem 1.1. The even degree part  $H^{even}(E, \mathbb{Z})$  of integral cohomology ring is generated by

$$y_1, y_2, b_2, \dots, b_{p-2}, C, v$$

with

$$\deg y_i = 2, \deg b_i = 2i$$

subject to the following relations:

$$\begin{aligned} py_i &= pb_j = pC = 0, \quad p^2 v = 0, \\ y_1 y_2^p - y_1^p y_2 &= 0, \\ y_i b_k &= b_k b_j = C b_j = 0, \\ y_i C &= y_i^p, \quad C^2 = y_1^{2p-2} + y_2^{2p-2} - y_1^{p-1} y_2^{p-1} \end{aligned}$$

by [10] or [8, Theorem 3] (see [13] also). In particular,  $p^2(H^{2n}(E, \mathbb{Z})) = 0$  for any  $n > 0$ .

On the other hand, the odd degree part of integral cohomology ring  $H^{odd}(E, \mathbb{Z})$  is annihilated by  $p$  and so it is considered as an  $\mathbb{F}_p[y_1, y_2, v]$ -module. As an  $\mathbb{F}_p[y_1, y_2, v]$ -module,  $H^{odd}(E, \mathbb{Z})$  is generated by two elements  $a_1$  and  $a_2$  with  $\deg a_i = 3$  subject to the following relations:

$$y_1 a_2 - y_2 a_1 = 0, \quad y_1^p a_2 - y_2^p a_1 = 0.$$

Let  $H^*(E, \mathbb{Z}) \rightarrow H^*(E, \mathbb{F}_p)$  be the natural map induced by  $\mathbb{Z} \rightarrow \mathbb{F}_p$ . We use the same letters for the images of  $y_i, b_j, C, v$  in  $H^*(E, \mathbb{F}_p)$ . Then

$$N = \mathbb{F}_p[v]\{b_2, \dots, b_{p-2}\} = \sqrt{0}$$

in

$$H^{even}(E, \mathbb{Z})/pH^{even}(E, \mathbb{Z}) = \mathbb{F}_p \otimes_{\mathbb{Z}} H^{even}(E, \mathbb{Z}).$$

Since

$$H^*(E) = (\mathbb{F}_p \otimes_{\mathbb{Z}} H^*(E, \mathbb{Z}))/\sqrt{0} = (\mathbb{F}_p \otimes_{\mathbb{Z}} H^{even}(E, \mathbb{Z}))/N,$$

there is a short exact sequence of  $A_p(E, E)$ -modules,

$$(3.1) \quad 0 \rightarrow N \rightarrow H^{even}(E, \mathbb{Z})/pH^{even}(E, \mathbb{Z}) \rightarrow H^*(E) \rightarrow 0.$$

On the other hand, since  $pH^{odd}(E, \mathbb{Z}) = 0$ , there is a short exact sequence of  $A_p(E, E)$ -modules,

$$(3.2) \quad 0 \longrightarrow H^{even}(E, \mathbb{Z})/pH^{even}(E, \mathbb{Z}) \longrightarrow H^{even}(E, \mathbb{F}_p) \longrightarrow H^{odd}(E, \mathbb{Z})[-1] \longrightarrow 0.$$

Let  $(y_1v, y_2v)H^{even}(E, \mathbb{Z})$  (resp.  $(y_1v, y_2v)H^*(E)$ ) be the ideal of  $H^{even}(E, \mathbb{Z})$  (resp.  $H^*(E)$ ) generated by  $y_1v$  and  $y_2v$ . Since  $py_i = 0$  and  $y_iN = 0$ , it follows that

$$(y_1v, y_2v)H^{even}(E, \mathbb{Z}) \cong (y_1v, y_2v)H^*(E).$$

Here we use Milnor's primitive operator  $Q_1 = P^1\beta - \beta P^1$  on  $H^*(-, \mathbb{F}_p)$ . This operator induces a map  $Q_1$  on  $H^*(-, \mathbb{Z})$  such that the following diagram commutes:

$$\begin{array}{ccc} H^{odd}(E, \mathbb{Z}) & \xrightarrow{Q_1} & H^{even}(E, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^{odd}(E, \mathbb{F}_p) & \xrightarrow{Q_1} & H^{even}(E, \mathbb{F}_p). \end{array}$$

Moreover,  $Q_1$  induces an isomorphism of  $A_p(E, E)$ -modules,

$$Q_1 : H^{odd}(E, \mathbb{Z}) \xrightarrow{\sim} (y_1v, y_2v)H^{even}(E, \mathbb{Z}) \cong (y_1v, y_2v)H^*(E)$$

$$Q_1(a_i) = y_iv,$$

(see [14, section 1]). Then we have

$$(3.3) \quad H^{odd}(E, \mathbb{Z}) \cong (y_1v, y_2v)H^*(E)[-2p+1]$$

and the proof of the first part of Theorem 1.1 is completed by the exact sequences (3.1) and (3.2).

Next we consider the odd degree part  $H^{odd}(E, \mathbb{F}_p)$ . Let  $K = \{x \in H^{even}(E, \mathbb{Z}) \mid px = 0\}$ . Then there exists a short exact sequence of  $A_p(E, E)$ -modules,

$$(3.4) \quad 0 \longrightarrow H^{odd}(E, \mathbb{Z}) \longrightarrow H^{odd}(E, \mathbb{F}_p) \longrightarrow K[-1] \longrightarrow 0.$$

Let  $H = H^{even}(E, \mathbb{Z}) \cap H^+(E, \mathbb{Z})$ . Since  $p^2H = 0$ ,  $pH \subset K$ . Moreover,  $K$ ,  $pH$  and  $H/K$  are  $A_p(E, E)$ -modules.

Since the map  $p : H \longrightarrow H$  is a homomorphism of  $A_{\mathbb{Z}}(E, E)$ -modules,

$$H/K \cong pH$$

as  $A_{\mathbb{Z}}(E, E)$ -modules. Since these are modules for  $A_p(E, E) = A_{\mathbb{Z}}(E, E)/pA_{\mathbb{Z}}(E, E)$ , these are isomorphic as  $A_p(E, E)$ -modules. Hence we have

$$K \leftrightarrow pH \oplus K/pH \leftrightarrow H/K \oplus K/pH \leftrightarrow H/pH.$$

Moreover, since

$$H/pH = \mathbb{F}_p \otimes H \leftrightarrow N \oplus H^+(E),$$

the proof of the second part of Theorem 1.1 is completed by (3.3) and (3.4).

Next we shall see the structure of  $N$ . Note that  $\text{res}_A^E(b_i) = 0$  for any  $i$  and any maximal elementary abelian  $p$ -subgroup  $A$  of  $E$ . Since the action of  $g \in \text{GL}_2(\mathbb{F}_p)$  is given by

$$g^*(b_i) = \det(g)^i b_i, \quad g^*(v) = \det(g)v$$

(see [8, Theorem 3]),  $N$  is a direct sum of simple  $A_p(E, E)$ -modules isomorphic to  $S(E, E, \det^i)$  for  $0 \leq i \leq p-2$ . Hence we have the following:

**Proposition 3.1.** *Let*

$$N_q = \mathbb{F}_p\{v^k b'_i \mid k \geq 0, k + i \equiv q \pmod{p-1}\}$$

for  $0 \leq q \leq p-2$ . Then

$$N = \bigoplus_{0 \leq q \leq p-2} N_q$$

and

$$N_q \cong \bigoplus S(E, E, \det^q)$$

as  $A_p(E, E)$ -modules.

Next, we shall consider the structure of the ideal  $(y_1 v, y_2 v)H^*(E)$ . Let  $I = (y_1 v, y_2 v)H^*(E)$ . Let  $\Gamma_S$  be the  $\mathbb{F}_p$ -subspace defined in Definition 2.2 for each simple  $A_p(E, E)$ -module  $S$ . We shall show that

$$Ie_S \cong I \cap \Gamma_S$$

for an idempotent  $e_S$  which corresponds to  $S$  and determine the  $\mathbb{F}_p$ -subspace  $I \cap \Gamma_S$  explicitly.

**Lemma 3.2.** *Let*

$$L = \mathbb{F}_p[y_1, y_2, C] \oplus \mathbb{F}_p[V]\{v, \dots, v^{p-2}, Cv, \dots, Cv^{p-2}\} \oplus \mathbb{F}_p[D_1]\{D_1\} \oplus \mathbb{F}_p[D_1]\{D_2\}$$

where  $\mathbb{F}_p[y_1, y_2, C]$  is the subalgebra of  $H^*(E)$  generated by  $y_1, y_2$  and  $C$ . Then

$$H^*(E) = I \oplus L.$$

*Proof.* Let  $(y_1, y_2, C)$  be the ideal of  $H^*(E)$  generated by  $y_1, y_2$  and  $C$ . Since

$$D_1 = C^p + V \equiv V \pmod{(y_1, y_2, C)},$$

we have

$$\begin{aligned} H^*(E) &= \mathbb{F}_p[v] \oplus (y_1, y_2, C) \\ &= \mathbb{F}_p \oplus \mathbb{F}_p[V]\{v, \dots, v^{p-2}\} \oplus \mathbb{F}_p[V]\{V\} \oplus (y_1, y_2, C) \\ &= \mathbb{F}_p \oplus \mathbb{F}_p[V]\{v, \dots, v^{p-2}\} \oplus \mathbb{F}_p[D_1]\{D_1\} \oplus (y_1, y_2, C). \end{aligned}$$

On the other hand, since

$$D_2 D_1 = CV(C^p + V) \equiv D_2 V \pmod{I},$$

it follows that

$$\begin{aligned} &\mathbb{F}_p \oplus (y_1, y_2, C) \\ &= \mathbb{F}_p[y_1, y_2, C] \oplus \mathbb{F}_p[v]\{Cv\} \oplus I \\ &= \mathbb{F}_p[y_1, y_2, C] \oplus \mathbb{F}_p[V]\{Cv, \dots, Cv^{p-2}\} \oplus \mathbb{F}_p[V]\{CV\} \oplus I \\ &= \mathbb{F}_p[y_1, y_2, C] \oplus \mathbb{F}_p[V]\{Cv, \dots, Cv^{p-2}\} \oplus \mathbb{F}_p[D_1]\{D_2\} \oplus I \end{aligned}$$

and we have  $H^*(E) = I \oplus L$ . □

**Lemma 3.3.** *For each simple  $A_p(E, E)$ -module  $S$ , we have*

$$\Gamma_S = (I \cap \Gamma_S) \oplus (L \cap \Gamma_S)$$



where  $I = (y_1v, y_2v)H^*(E)$  and  $L$  is an  $\mathbb{F}_p$ -subspace defined in Lemma 3.2. Moreover,

(1) If  $S = S(E, C_p, U_i)$ , then  $I \cap \Gamma_S = 0$ .

(2) If  $S = S(E, A, S^{p-1} \otimes \det^q)$ , then

$$I \cap \Gamma_S = \Gamma_S = \begin{cases} \mathbb{DA}\{\bigoplus_{0 \leq j \leq p-1} D_2 C^j (\mathbb{F}_p C + S^{p-1})\} & (q = 0) \\ \mathbb{DA}\{\bigoplus_{0 \leq j \leq p-1} v^q C^j (CS^q + T^q)\} & (1 \leq q \leq p-2). \end{cases}$$

(3)

$$I \cap \Gamma_S = \begin{cases} \mathbb{DA}\{D_2^2\} & (S = S(E, E, S^0)) \\ \mathbb{CA}\{C^2 v^q\} & (S = S(E, E, \det^q), 1 \leq q \leq p-2) \\ \mathbb{DA}\{V S^{p-1}\} & (S = S(E, E, S^{p-1})) \\ \mathbb{CA}\{v^q S^{p-1}\} & (S = S(E, E, S^{p-1} \otimes \det^q), 1 \leq q \leq p-2) \end{cases}$$

(4) Let

$$S = S^i v^q, \quad T = T^{p-i-1} v^s$$

for  $1 \leq i \leq p-2$ ,  $0 \leq q \leq p-2$ , where  $s \equiv i+q \pmod{p-1}$ ,  $0 \leq s \leq p-2$ . Then  $I \cap \Gamma_{S(E, E, S^i \otimes \det^q)}$  is the following  $\mathbb{F}_p$ -subspace:

$$\begin{aligned} \mathbb{CA}\{VS\} &\oplus \mathbb{DA}\{VT\} & (q \equiv 2i \equiv 0) \\ \mathbb{CA}\{VS\} &\oplus \mathbb{CA}\{T\} & (q \equiv 0, 2i \not\equiv 0) \\ \mathbb{DA}\{S\} &\oplus \mathbb{DA}\{VT\} & (i = q, 3i \equiv 0) \\ \mathbb{DA}\{S\} &\oplus \mathbb{CA}\{T\} & (i = q, 3i \not\equiv 0, 2i \not\equiv 0) \\ \mathbb{DA}\{S\} &\oplus \mathbb{CA}\{VT\} & (i = q, 3i \not\equiv 0, 2i \equiv 0) \\ \mathbb{CA}\{S\} &\oplus \mathbb{DA}\{VT\} & (q \not\equiv 0, i \not\equiv q, q+2i \equiv 0) \\ \mathbb{CA}\{S\} &\oplus \mathbb{CA}\{T\} & (q \not\equiv 0, i \not\equiv q, q+2i \not\equiv 0, i+q \not\equiv 0) \\ \mathbb{CA}\{S\} &\oplus \mathbb{CA}\{VT\} & (q \not\equiv 0, i \not\equiv q, q+2i \not\equiv 0, i+q \equiv 0). \end{aligned}$$

(5)  $I \cap \Gamma_{S(1,1,\mathbb{F}_p)} = 0$ .

*Proof.* (1) If  $S = S(E, C_p, U_i)$  ( $0 \leq i \leq p-2$ ), then  $I \cap \Gamma_S = 0$  and  $\Gamma_S \subset L$  by Definition 2.2.

(2) If  $S = S(E, A, S^{p-1} \otimes \det^q)$  ( $0 \leq q \leq p-2$ ), then  $\Gamma_S \subset I$  since  $D_2 C = C^2 V \in I$ .

(3) If  $S = S(E, E, S^0)$ , then  $\Gamma_S = \mathbb{DA}^+$ ,

$$\mathbb{DA}^+ = \mathbb{DA}\{D_2^2\} \oplus \mathbb{F}_p[D_1]\{D_1, D_2\}.$$

Since  $\mathbb{DA}\{D_2^2\} \subset I$  and  $\mathbb{F}_p[D_1]\{D_1, D_2\} \subset L$ , we have

$$\mathbb{DA}^+ = (I \cap \Gamma_S) \oplus (L \cap \Gamma_S)$$

and  $I \cap \Gamma_S = \mathbb{DA}\{D_2^2\}$ .

If  $S = S(E, E, \det^q)$  ( $1 \leq q \leq p-2$ ), then  $\Gamma_S = \mathbb{CA}\{v^q\}$ ,

$$\mathbb{CA}\{v^q\} = \mathbb{CA}\{C^2 v^q\} \oplus \mathbb{F}_p[V]\{v^q, C v^q\}.$$

Since  $C^2 v^q \in I$  and  $\mathbb{F}_p[V]\{v^q, C v^q\} \subset L$ , it follows that

$$\mathbb{CA}\{v^q\} = (I \cap \Gamma_S) \oplus (L \cap \Gamma_S)$$

and  $I \cap \Gamma_S = \mathbb{CA}\{C^2 v^q\}$ .

If  $S = S(E, E, S^{p-1})$ , then  $\Gamma_S = \mathbb{DA}\{V S^{p-1}\}$ . If  $S = S(E, E, S^{p-1} \otimes \det^q)$  ( $1 \leq q \leq p-2$ ),  $\Gamma_S = \mathbb{CA}\{v^q S^{p-1}\}$ . In these cases,  $\Gamma_S \subset I$ .

(4) Let  $S = S^i v^q$  ( $1 \leq i \leq p-2$ ,  $0 \leq q \leq p-2$ ). If  $q = 0$  then  $SV \subset I$ . If  $q \neq 0$  then  $S \subset I$ . Hence the first term of  $\Gamma_S$  is contained in  $I$ .

Let  $T = T^{p-i-1}v^s$ ,  $s \equiv i + q \pmod{p-1}$ ,  $0 \leq s \leq p-2$ . If  $s \equiv 0$ , then  $VT \subset I$ . If  $i + q \not\equiv 0$ , then  $T \subset I$ . Hence the second term of  $\Gamma_S$  is contained in  $I$  unless  $i = q$ ,  $3i \not\equiv 0$ ,  $2i \equiv 0$ , or  $q \neq 0$ ,  $i \neq q$ ,  $q + 2i \not\equiv 0$ ,  $i + q \equiv 0$ . In these cases,

$$\mathbb{CA}\{T\} = \mathbb{F}_p[C]\{T\} \oplus \mathbb{CA}\{VT\}$$

where  $\mathbb{CA}\{VT\} \subset I$ ,  $\mathbb{F}_p[C]\{T\} \subset L$ . Hence

$$\mathbb{CA}\{T\} = \mathbb{CA}\{T\} \cap I \oplus \mathbb{F}_p[C]\{T\} \cap I$$

and  $\mathbb{CA}\{T\} \cap I = \mathbb{CA}\{VT\}$ . □

**Lemma 3.4.** *Let  $I = (y_1v, y_2v)H^*(E)$ . Then*

$$I = \bigoplus_S (I \cap \Gamma_S).$$

*Proof.* First,  $H^*(E) = \bigoplus_S \Gamma_S = I \oplus L$  by Proposition 2.5 and Lemma 3.2. On the other hand,

$$\Gamma_S = (I \cap \Gamma_S) \oplus (L \cap \Gamma_S)$$

by Lemma 3.3. Hence

$$\begin{aligned} H^*(E) &= \bigoplus_S \Gamma_S = \bigoplus_S ((I \cap \Gamma_S) \oplus (L \cap \Gamma_S)) \\ &= (\bigoplus_S (I \cap \Gamma_S)) \oplus (\bigoplus_S (L \cap \Gamma_S)) \subset I \oplus L = H^*(E). \end{aligned}$$

Hence we have

$$I = \bigoplus_S (I \cap \Gamma_S).$$

□

Now, we determine the  $\mathbb{F}_p$ -vector space  $Ie_S$  for any simple  $A_p(E, E)$ -module  $S$ .

**Theorem 3.5.** *Let  $S$  be a simple  $A_p(E, E)$ -module. Then there exists an idempotent  $e_S$  corresponding to  $S$  such that*

$$Ie_S = (I \cap \Gamma_S)e_S \cong I \cap \Gamma_S.$$

*Proof.* By Theorem 2.3,  $\Gamma_S \cong \Gamma_S e_S$  for some idempotent corresponding to  $S$ . Hence  $e_S$  induces an isomorphism

$$I \cap \Gamma_S \cong (I \cap \Gamma_S)e_S.$$

For a graded  $\mathbb{F}_p$ -subspace  $M \subset H^*(E)$ , let  $M^n = H^n(E) \cap M$ . Then

$$\begin{aligned} \dim I^n &= \sum_S \dim(I^n)e_S \geq \sum_S \dim(I^n \cap \Gamma_S)e_S \\ &= \sum_S \dim I^n \cap \Gamma_S = \dim I^n. \end{aligned}$$

The last equality follows from Lemma 3.4. Hence we have

$$Ie_S = (I \cap \Gamma_S)e_S.$$

□

**Remark 3.6.** Let  $X$  be the indecomposable summand in the complete stable splitting of  $BE$  which corresponds to a simple  $A_p(E, E)$ -modules  $S$ . Let  $e_S$  be an idempotent which correspond to  $S$  as above. Then we can get the  $\mathbb{F}_p$ -vector space

$$H^*(E, \mathbb{F}_p)e_S \cong H^*(\vee X, \mathbb{F}_p) \cong \bigoplus^d H^*(X, \mathbb{F}_p)$$

where  $\vee X$  is a wedge sum of  $d = \dim S$  copies of  $X$ , from Theorem 1.1, Proposition 3.1 and Theorem 3.5.

**Corollary 3.7.** *Every simple  $A_p(E, E)$ -module appears as a composition factor in  $H^{2n}(E, \mathbb{F}_p)$  for some  $n \leq p^2 - 2$ .*

*Proof.* Let  $H(p^2 - 2) = \bigoplus_{n=0}^{p^2-2} H^{2n}(E, \mathbb{F}_p)$ . For a simple  $A_p(E, E)$ -module  $S$ , let  $2\gamma(S)$  be the lowest degree such that  $(\Gamma_S \cap I)^{2\gamma(S)} \neq 0$ . If  $2\gamma(S) \leq 2(p+2)(p-1)$ , then  $2(\gamma(S) - p) \leq 2(p^2 - 2)$ . Since  $S$  appears in the degree  $2(\gamma(S) - p)$  part of  $I[-2p]$ , it follows that  $S$  appears in  $H^{2(\gamma(S)-p)}(E, \mathbb{F}_p)$  by Theorem 1.1. Hence  $S$  appears in  $H(p^2 - 2)$ . In particular, if  $\Gamma_S \leq I$ , then  $S$  appears in  $H(p^2 - 2)$  by Corollary 2.4. This implies that simple modules

$$S(E, A, S(A)^{p-1} \otimes \det^q) (0 \leq q \leq p-2), \quad S(E, E, S^{p-1} \otimes \det^q) (0 \leq q \leq p-2)$$

appear in  $H(p^2 - 2)$ .

On the other hand, since the degrees of  $C^2 v^q$ ,  $(1 \leq q \leq p-2)$  and  $VS^i$ ,  $S^i v^q$ ,  $(1 \leq i \leq p-2, 0 \leq q \leq p-2)$  are all smaller than  $\deg D_2 S^{p-1} = 2(p+2)(p-1)$ , we have that  $S(E, E, \det^q)$ ,  $(1 \leq q \leq p-2)$  and  $S(E, E, S^i \otimes \det^q)$ ,  $(1 \leq i \leq p-2, 0 \leq q \leq p-2)$  appear in  $H(p^2 - 2)$ .

Moreover, since  $S(E, C_p, U_i)$   $(0 \leq i \leq p-2)$  appears in  $H^2(E) \oplus \cdots \oplus H^{2(p-1)}(E)$ , it appears in  $H(p^2 - 2)$ . Finally, we consider  $S(E, E, S^0)$ . Since it appears in  $H^{2p(p-1)}(E)$  and  $2p(p-1) (= \deg D_1) \leq 2(p^2 - 2)$ ,  $S(E, E, S^0)$  appears in  $H(p^2 - 2)$ . This completes the proof.  $\square$

#### 4. STABLE SPLITTING OF GROUPS RELATED TO $L_3(p)$

In this section, we consider the stable splitting of  $BG$  for groups  $G$  having  $E$  as a Sylow  $p$ -subgroup, in particular the linear group  $L_3(p)$  and its extensions.

Benson and Fechtbach [2], Martino and Priddy [11] prove the following theorem on complete stable splitting. Let  $P$  be a finite  $p$ -group. If  $G$  is a finite group which contains  $E$ , then  $G$  is considered as an  $(E, E)$ -biset. We denote by  $[G]$  the element of  $A_p(E, E)$  corresponding to  $G$ . Let  $S(P, Q, V)$  be the simple  $A_p(P, P)$ -module which corresponds to  $(Q, V)$  where  $Q$  is a subgroup of  $P$  and  $V$  is a simple  $\mathbb{F}_p \text{Out}(P)$ -module.

**Theorem 4.1** ([2], [11]). *Let  $G$  be a finite group with Sylow  $p$ -subgroup  $P$ . The complete stable splitting of  $BG$  is given by*

$$BG \sim \bigvee_{(Q,V)} \dim(S(G, Q, V)) X_{S(P,Q,V)}$$

where  $S(G, Q, V) = S(P, Q, V)[G]$ .

Let

$$H^*(G) = (\mathbb{F}_p \otimes H^*(G, \mathbb{Z})) / \sqrt{(0)}$$

for a finite group  $G$ . From Corollary 2.4 and Corollary 3.7, we have the following.

**Corollary 4.2.** *Let  $G_1, G_2$  have the same  $p$ -Sylow subgroup  $E$ . Suppose that  $G_1 \leq G_2$ .*

(1) *If*

$$\dim H^{2n}(G_1) = \dim H^{2n}(G_2)$$

*for all  $0 \leq n \leq (p+2)(p-1)$ , then  $BG_1 \sim BG_2$ .*

(2) *If*

$$\dim H^{2n}(G_1, \mathbb{F}_p) = \dim H^{2n}(G_2, \mathbb{F}_p)$$

*for all  $0 \leq n \leq p^2 - 2$ , then  $BG_1 \sim BG_2$ .*

In general, the computation of these  $\dim S(G, Q, V)$  is not so easy. Hence we study the way to compute it from the information on the cohomology  $H^*(G)$ . (In fact, in [15], most direct summands in the stable splitting of  $BG$  are computed from  $H^*(G)$ .)

Let  $\mathcal{F}_G$  be the fusion system on  $E$  determined by  $G$ . Let  $\mathcal{F}_G^{ec}$ -rad be the set of  $\mathcal{F}_G^{ec}$ -radical maximal elementary abelian  $p$ -subgroups of  $E$ . If  $A$  is a maximal elementary abelian  $p$ -subgroup of  $E$ , then  $A \in \mathcal{F}_G^{ec}$ -rad if and only if  $W_G(A) = N_G(A)/C_G(A) = \text{Out}_{\mathcal{F}_G}(A) \geq SL_2(\mathbb{F}_p)$  by [12, Lemma 4.1]. Let  $W_G(E) = N_G(E)/EC_G(E) = \text{Out}_{\mathcal{F}_G}(E)$ .

**Theorem 4.3** ([13, Theorem 4.3], [15, Theorem 3.1]). *Let  $G$  have the Sylow  $p$ -subgroup  $E$ , then*

$$H^*(G) \cong H^*(E)[G] = H^*(E)^{W_G(E)} \cap (\cap_{A \in \mathcal{F}_G^{ec}\text{-rad}} (\text{res}_A^E)^{-1}(H^*(A)^{W_G(A)})).$$

*Moreover, if  $M$  is an  $A_p(E, E)$ -submodule of  $H^*(E)$ , then*

$$M[G] = M^{W_G(E)} \cap (\cap_{A \in \mathcal{F}_G^{ec}\text{-rad}} (\text{res}_A^E)^{-1}(H^*(A)^{W_G(A)})).$$

*Proof.* The first part follows from Alperin's fusion theorem ([3, Theorem A.10]). Let  $M$  be an  $A_p(E, E)$ -submodule of  $H^*(E)$ . Since  $[G][G] \in \mathbb{F}_p[G]$ , it follows that  $M[G] = M \cap H^*(E)[G]$ . Hence the result follows from the first part.  $\square$

Let  $X_{i,q}$  be the indecomposable summand in the stable splitting of  $BE$  which corresponds to the simple  $A_p(E, E)$ -module  $S(E, E, S^i \otimes \det^q)$ . For  $0 \leq q \leq p-2$ , let  $L(2, q)$  (resp.  $L(1, q)$ ) be the summand which corresponds to the simple  $A_p(E, E)$ -module  $S(E, A, S(A)^{p-1} \otimes \det^q)$  (resp.  $S(E, C_p, U_q)$ ). We set  $M(2) = L(1, 0) \vee L(2, 0)$ .

Suppose that the stable splitting of  $BG$  is written as

$$BG \sim (\vee_{i,q} n(G)_{i,q} X_{i,q}) \vee (\vee_q m(G, 2)_q L(2, q)) \vee (\vee_q m(G, 1)_q L(1, q)).$$

Recall that

$$H^{2q}(E) \cong \begin{cases} S(E, C_p, U_i) & (1 \leq q \leq p-2) \\ S(E, C_p, U_0) & (q = p-1) \end{cases}$$

by Theorem 2.3. Hence,

**Lemma 4.4** ([15, Corollary 4.6]). *The multiplicity  $m(G, 1)_q$  for  $L(1, q)$  is given by*

$$m(G, 1)_q = \begin{cases} \dim H^{2q}(G) & (1 \leq q \leq p-2) \\ \dim H^{2(p-1)}(G) & (q = 0). \end{cases}$$

The multiplicity  $n(G)_{i,q}$  of  $X_{i,q}$  depends only on  $W_G(E) = N_G(E)/EC_G(E)$ . For  $H \leq GL_2(\mathbb{F}_p)$  and  $GL_2(\mathbb{F}_p)$ -submodule  $M$  of  $H^*(E)$ , let

$$M^H = \{m \in M \mid mh = m \text{ for any } h \in H\}$$

denotes the subspace consists of  $H$ -invariant elements. Then we have the following lemma.

**Lemma 4.5** ([15, Lemma 4.7]). *The multiplicity  $n(G)_{i,q}$  of  $X_{i,q}$  in  $BG$  is given by*

$$n(G)_{i,q} = \dim(S^i v^q)^{W_G(E)}.$$

Next problem is to seek the multiplicity  $m(G, 2)_q$  for the summand  $L(2, q)$  in  $BG$ . We can prove,

**Lemma 4.6** ([15, Proposition 4.9]). *The multiplicity of  $L(2, 0)$  in  $BG$  is given by*

$$m(G, 2)_0 = \sharp_G(A) - \sharp_G(F^{ec}A)$$

where  $\sharp_G(A)$  (resp.  $\sharp_G(F^{ec}A)$ ) is the number of  $G$ -conjugacy classes of rank two elementary abelian  $p$ -subgroups in  $E$  (resp. subgroups in  $\mathcal{F}_G^{ec}\text{-rad}$ ).

**Lemma 4.7** ([15, Corollary 4.10]). *The multiplicity of  $L(1, 0)$  in  $BG$  is given by*

$$m(G, 1)_0 = \dim H^{2(p-1)}(G) = \sharp_G(A) - \sharp_G(F^{ec}A).$$

**Remark 4.8.** By Lemma 4.6 and 4.7,  $m(G, 1)_0 = m(G, 2)_0$ , namely,  $L(1, 0)$  and  $L(2, 0)$  always appear in  $BG$  as  $M(2) = L(1, 0) \vee L(2, 0)$ . On the other hand, in Corollary 2.4, all simple modules except for  $S(E, A, S^{p-1})$  appear in  $H^{2n}(E)$  for  $n \leq p^2 - 1$ . Note that the minimal  $n$  such that  $S(E, E, S^{p-1})$  appears in  $H^{2n}(E)$  is  $p^2 - 1 = \frac{1}{2} \deg(VS^{p-1})$ . Hence we may replace the bound  $(p+2)(p-1)$  by  $p^2 - 1$  in Corollary 4.2 (1).

For the number  $m(G, 2)_q$  for  $q \neq 0$ , it seems that there is not a good way to find it. However we give some condition such that  $m(G, 2)_q = 0$ .

**Lemma 4.9** ([15, Lemma 4.11]). *Let  $\xi \in \mathbb{F}_p^*$  be a primitive  $(p-1)$ -th root of 1. Suppose that  $G \supset E: \langle \text{diag}(\xi, \xi) \rangle$ . If  $\xi^{3k} \neq 1$ , then  $BG$  does not contain the summand  $L(2, k)$ , i.e.,  $m(G, 2)_k = 0$ .*

Let  $\xi$  be the multiplicative generator of  $\mathbb{F}_p^*$  as above. Let

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2(\mathbb{F}_p).$$

Let

$$T = \langle \text{diag}(\xi, \xi), \text{diag}(\xi, 1) \rangle$$

be a torus in  $\text{GL}_2(\mathbb{F}_p)$ . Then  $w$  normalizes  $T$ . Let  $T\langle w \rangle$  be the semidirect product of  $T$  by  $\langle w \rangle$

**Lemma 4.10.** *Assume that  $1 \leq l \leq p-1$ ,  $0 \leq k \leq p-2$ . Then*

$$(S^l v^k)^T = \begin{cases} \mathbb{F}_p y_1^i y_2^i v^{p-1-i} & (l = 2i, \ k = p-1-i, \ 1 \leq i \leq \frac{p-1}{2}) \\ \mathbb{F}_p y_1^{p-1} \oplus \mathbb{F}_p y_2^{p-1} & (l = p-1, \ k = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$(S^l v^k)^{T\langle w \rangle} = \begin{cases} \mathbb{F}_p y_1^{2j} y_2^{2j} v^{p-1-2j} & (l = 4j, \ k = p-1-2j, \ 1 \leq j \leq \frac{p-1}{4}) \\ \mathbb{F}_p (y_1^{p-1} + y_2^{p-1}) & (l = p-1, \ k = 0) \\ 0 & (\text{otherwise}). \end{cases}$$

*Proof.* We consider the action of  $\text{diag}(\xi, 1)$  and  $\text{diag}(1, \xi)$ . For  $0 \leq i \leq l$ , we have

$$\begin{aligned}\text{diag}(\xi, 1)y_1^i y_2^{l-i} v^k &= \xi^{i+k} y_1^i y_2^{l-i} v^k \\ \text{diag}(1, \xi)y_1^i y_2^{l-i} v^k &= \xi^{l-i+k} y_1^i y_2^{l-i} v^k.\end{aligned}$$

Hence  $y_1^i y_2^{l-i} v^k$  is  $T$ -invariant if and only if  $i+k \equiv l-i+k \equiv 0 \pmod{p-1}$ , namely,  $l \equiv 2i \pmod{p-1}$  and  $i+k \equiv 0 \pmod{p-1}$ . If  $k=0$ , then  $i=0, p-1, l=p-1$ . If  $k>0$ , then  $i=p-1-k, l=2i$ . Since  $1 \leq l \leq p-1$ , it follows that  $i \leq \frac{p-1}{2}$ .

Next consider the action of  $w$ . Since  $w$  interchanges  $y_1$  and  $y_2$ ,  $wv = -v$ ,  $y_1^i y_2^j v^{p-1-i}$  is  $w$ -invariant if and only if  $i$  is even.  $\square$

Now assume that  $p-1 = 3m$ . Note that  $m$  is even. We set

$$H = \langle \text{diag}(\xi, \xi), \text{diag}(\xi^3, 1) \rangle.$$

Then  $w$  normalizes  $H$ . Let  $H\langle w \rangle$  be the semidirect product of  $H$  by  $\langle w \rangle$ .

**Lemma 4.11.** *Assume that  $p-1 = 3m$ . Let  $m = 2n$ . Assume that  $1 \leq l \leq p-1$ ,  $0 \leq k \leq p-2$ . Then  $(S^l v^k)^H$  is equal to the following vector space.*

$$\begin{cases} \mathbb{F}_p\{y_1^{p-1}, y_1^{2m} y_2^m, y_1^m y_2^{2m}, y_2^{p-1}\} & (l = p-1, k = 0) \\ \mathbb{F}_p\{y_1^{i-n} y_2^{i+n} v^{3n-i}, y_1^{i+n} y_2^{i-n} v^{3n-i}\} & (l = 2i, k = 3n-i, n \leq i < 3n) \\ \mathbb{F}_p\{y_1^i y_2^i v^{3m-i}\} & (l = 2i, k = (p-1) - i, 1 \leq i < m) \\ \mathbb{F}_p\{y_1^{i-m} y_2^{i+m} v^{3m-i}, y_1^i y_2^i v^{3m-i}, y_1^{i+m} y_2^{i-m} v^{3m-i}\} & (l = 2i, k = (p-1) - i, m \leq i \leq 3n) \\ 0 & (\text{otherwise}). \end{cases}$$

*Proof.* We consider the action of  $\text{diag}(\xi, \xi)$  and  $\text{diag}(\xi^3, 1)$  on  $y_1^j y_2^{l-j} v^k$  for  $0 \leq j \leq l$ . Since

$$\text{diag}(\xi, \xi)y_1^j y_2^{l-j} v^k = \xi^{l+2k} y_1^j y_2^{l-j} v^k$$

and

$$\text{diag}(\xi^3, 1)y_1^j y_2^{l-j} v^k = \xi^{3j+3k} y_1^j y_2^{l-j} v^k,$$

$y_1^j y_2^{l-j} v^k$  is  $H$ -invariant if and only if

$$\begin{cases} l+2k \equiv 0 & \pmod{p-1} \\ j+k \equiv 0 & \pmod{m}. \end{cases}$$

Note that the condition  $l+2k \equiv 0 \pmod{p-1}$  implies that  $l$  is even, so we set  $l = 2i$ ,  $1 \leq i \leq 3n$ . Then  $y_1^j y_2^{l-j} v^k \in (S^l v^k)^H$  if and only if

$$\begin{cases} i+k \equiv 0 & \pmod{3n} \\ j+k \equiv 0 & \pmod{m}. \end{cases}$$

Since  $0 \leq k \leq p-2$ ,  $k \equiv -i \pmod{3n}$  if and only if

$$k = 3n - i \text{ or } (p-1) - i.$$

First, assume  $k = 3n - i$ . Then  $j \equiv -k = i - 3n \pmod{m}$  if and only if  $j = i + sn$  for some integer  $s \in \mathbb{Z}$  such that  $s \equiv 1 \pmod{2}$ . Then

$$\begin{aligned} 0 \leq j \leq l = 2i &\iff 0 \leq i + sn \leq 2i \\ &\iff -i \leq sn \leq i \\ &\iff |s| \leq \frac{i}{n}. \end{aligned}$$

If  $1 \leq i < n$ , namely,  $\frac{i}{n} < 1$ , then there is no  $s \in \mathbb{Z}$  such that  $|s| \leq \frac{i}{n}$  with  $s \equiv 1 \pmod{2}$ .

Assume that  $n \leq i \leq 3n$ . If  $i = 3n$ , namely,  $l = p - 1$  and  $k = 0$ , then, since  $\frac{i}{n} = 3$ ,  $|s| \leq \frac{i}{n}$  if and only if  $s = -3, -1, 1, 3$ . Then

$$j = i + sn = (3 + s)n = 0, m, 2m, 3m.$$

If  $n \leq i < 3n$ , namely,  $1 \leq \frac{i}{n} < 3$ , then  $|s| \leq \frac{i}{n}$  if and only if  $s = -1, 1$  since  $s \equiv 1 \pmod{2}$ . So we have

$$j = i + sn = i \pm n.$$

Next we consider the case  $k = (p - 1) - i$ . Then,

$$j \equiv -k = i - (p - 1) \pmod{m}$$

if and only if  $j = i + sm$  for some  $s \in \mathbb{Z}$ . Then,

$$\begin{aligned} 0 \leq j = i + sm \leq l = 2i &\iff -i \leq sm \leq i \\ &\iff |s| \leq \frac{i}{m}. \end{aligned}$$

If  $1 \leq i < m$ , then this implies  $s = 0$  and  $j = i$ . If  $m \leq i \leq 3n$ , namely,  $1 \leq \frac{i}{m} \leq \frac{3}{2} < 2$ , then this implies  $s = 0, \pm 1$ . Hence we have  $j = i, i \pm m$ . This completes the proof.  $\square$

**Lemma 4.12.** Assume that  $p - 1 = 3m$ . Let  $m = 2n$ . Assume that  $1 \leq l \leq p - 1$ ,  $0 \leq k \leq p - 2$ . Then  $(S^l v^k)^{H(w)}$  is equal to the following vector space.

$$\begin{cases} \mathbb{F}_p\{y_1^{p-1} + y_2^{p-1}, y_1^{2m}y_2^m + y_1^m y_2^{2m}\} & (l = p - 1, k = 0) \\ \mathbb{F}_p\{(y_1^{i-n}y_2^{i+n} + (-1)^{3n-i}y_1^{i+n}y_2^{i-n})v^{3n-i}\} & (l = 2i, k = 3n - i, n \leq i < 3n) \\ \mathbb{F}_p\{y_1^i y_2^i v^{3m-i}\} & (l = 2i, k = (p - 1) - i, 1 \leq i < m, i: \text{even}) \\ \mathbb{F}_p\{(y_1^{i-m}y_2^{i+m} + y_1^{i+m}y_2^{i-m})v^{3m-i}, y_1^i y_2^i v^{3m-i}\} & (l = 2i, k = (p - 1) - i, m \leq i \leq 3n, i: \text{even}) \\ \mathbb{F}_p\{(y_1^{i-m}y_2^{i+m} - y_1^{i+m}y_2^{i-m})v^{3m-i}\} & (l = 2i, k = (p - 1) - i, m \leq i \leq 3n, i: \text{odd}) \\ 0 & (\text{otherwise}). \end{cases}$$

*Proof.* Since  $w$  interchanges  $y_1$  and  $y_2$ ,  $vw = (\det w)v = -v$ , this lemma follows from the previous lemma.  $\square$

Let  $p - 1 = 3m$ . We consider the multiplicity  $m(G, 2)_m$  and  $m(G, 2)_{2m}$  in some cases. Recall that  $(CS^q + T^q)v^q \cong S(E, A, S(A)^{p-1} \otimes \det^q)$  for  $1 \leq q \leq p - 2$  by [7, Corollary 9.3]. This module has a basis

$$(y_1^i y_2^j) v^q, (i = 0, q \leq i \leq p - 1 + q, j = p - 1 + q - i).$$

Note that

$$y_1^i y_2^j, (i = 0, p \leq i \leq p - 1 + q, j = p - 1 + q - i)$$

is a basis of  $CS^q$ . On the other hand,

$$y_1^i y_2^j, (q \leq i \leq p - 1, j = p - 1 + q - i)$$

is a basis of  $T^q$ . Moreover, if  $q = m$  or  $q = 2m$ , then all elements in  $(CS^q + T^q)v^q$  are  $\text{diag}(\xi, \xi)$ -invariant since

$$\text{diag}(\xi, \xi)(y_1^i y_2^j v^q) = \xi^{i+j+2q}(y_1^i y_2^j v^q) = \xi^{p-1+3q}(y_1^i y_2^j v^q) = (y_1^i y_2^j v^q).$$

**Lemma 4.13.** *Let  $M = (CS^m + T^m)v^m$ .*

(1)  $M^T$  has a basis

$$y_1^{2m} y_2^{2m} v^m.$$

(2)  $M^H$  has a basis

$$y_1^{4m} v^m, y_1^{3m} y_2^m v^m, y_1^{2m} y_2^{2m} v^m, y_1^m y_2^{3m} v^m, y_2^{4m} v^m.$$

(3)  $M^{T\langle w \rangle}$  has a basis

$$y_1^{2m} y_2^{2m} v^m.$$

(4)  $M^{H\langle w \rangle}$  has a basis

$$(y_1^{4m} + y_2^{4m})v^m = C(y_1^m + y_2^m)v^m, y_1^m y_2^m (y_1^{2m} + y_2^{2m})v^m, y_1^{2m} y_2^{2m} v^m.$$

*Proof.* Since

$$\text{diag}(\xi, 1)(y_1^i y_2^j v^m) = \xi^{i+m}(y_1^i y_2^j v^m)$$

and

$$\text{diag}(\xi^3, 1)(y_1^i y_2^j v^m) = \xi^{3(i+m)}(y_1^i y_2^j v^m) = \xi^{3i}(y_1^i y_2^j v^m),$$

$y_1^i y_2^j v^m$  is  $T$ -invariant if and only if  $i+m \equiv 0 \pmod{p-1}$ . Moreover,  $y_1^i y_2^j v^m$  is  $H$ -invariant if  $i \equiv 0 \pmod{m}$ .

(1) Since  $i = 0$  or  $m \leq i \leq p-1+m = 4m$ ,  $i+m \equiv 0 \pmod{p-1}$  if and only if  $i = 2m$ .

(2) Since  $i = 0$  or  $m \leq i \leq p-1+m = 4m$ ,  $i \equiv 0 \pmod{m}$  if and only if  $i = 0, m, 2m, 3m, 4m$ .

(3) (4) Since  $m$  is even,  $w$  acts on  $v^m$  trivially. On the other hand  $w$  interchanges  $y_1$  and  $y_2$ . Hence the results follows from (1) and (2).  $\square$

Similarly, we have the following.

**Lemma 4.14.** *Let  $M = (CS^{2m} + T^{2m})v^{2m}$ .*

(1)  $M^T$  has a basis

$$C y_1^m y_2^m v^{2m} = y_1^{4m} y_2^m v^{2m}.$$

(2)  $M^H$  has a basis

$$y_1^{5m} v^{2m}, C y_1^m y_2^m v^{2m} = y_1^{4m} y_2^m v^{2m}, y_1^{3m} y_2^{2m} v^{2m}, y_1^{2m} y_2^{3m} v^{2m}, y_2^{5m} v^{2m}.$$

(3)  $M^{T\langle w \rangle}$  has a basis

$$C y_1^m y_2^m v^{2m} = y_1^{4m} y_2^m v^{2m}.$$

(4)  $M^{H\langle w \rangle}$  has a basis

$$(y_1^{5m} + y_2^{5m})v^{2m} = C(y_1^{2m} + y_2^{2m})v^{2m}, y_1^{2m} y_2^{2m} (y_1^m + y_2^m)v^{2m}, C y_1^m y_2^m v^{2m}.$$

If  $A \in \mathcal{A}(E)$  is a maximal elementary abelian  $p$ -subgroup of  $E$ , then

$$H^*(A) = \mathbb{F}_p[y_A, u_A], \deg y_A = \deg u_A = 2.$$

We may assume that

$$\text{res}_{A_i}^E(y_1) = y_{A_i}, \text{res}_{A_i}^E(y_2) = i y_{A_i} \text{ for } i \in \mathbb{F}_p$$

and

$$\text{res}_{A_\infty}^E(y_1) = 0, \text{res}_{A_\infty}^E(y_2) = y_{A_\infty}.$$

Moreover,

$$\text{res}_A^E(C) = y_A^{p-1}, \text{res}_A^E(v) = u_A^p - y_A^{p-1} u$$

for any  $A \in \mathcal{A}(E)$  (see [7, section 4]).



**Lemma 4.15.** *Let  $1 \leq q \leq p-2$ . Then*

$$((CS^q + T^q)v^q)[G] = ((CS^q + T^q)v^q)^{W_G(E)} \cap (\cap_{A \in \mathcal{F}_G^{ec}\text{-rad}} \ker \text{res}_A^E)$$

*Proof.* Let  $y = y_A$  and  $u = u_A$  for  $A \in \mathcal{A}(E)$ . Then

$$\text{res}_A^E((CS^q + T^q)v^q) = \mathbb{F}_p y^{p-1+q} \text{res}_A^E(v^q) = \mathbb{F}_p y^{p-1}(yu^p - y^p u)^q.$$

If  $g \in \text{Aut}(A) = \text{GL}_2(\mathbb{F}_p)$ , then

$$g(yu^p - y^p u) = (\det g)(yu^p - y^p u)$$

and  $y^{p-1}(yu^p - y^p u)^q$  is not  $\text{SL}_2(\mathbb{F}_p)$ -invariant, hence the result follows from Theorem 4.3.  $\square$

**Proposition 4.16.** *Suppose that  $\mathcal{F}_G^{ec}\text{-rad} = \{A_0, A_\infty\}$ . Then  $m(G, 2)_m = m(G, 2)_{2m}$  and we have the following values:*

$W_G(E)$	$H$	$H\langle w \rangle$	$T$	$T\langle w \rangle$
$m(G, 2)_m = m(G, 2)_{2m}$	3	2	1	1

*Proof.* Since

$$\text{res}_{A_0}^E(y_1) \neq 0, \quad \text{res}_{A_0}^E(y_2) = 0$$

and

$$\text{res}_{A_\infty}^E(y_1) = 0, \quad \text{res}_{A_\infty}^E(y_2) \neq 0,$$

the results follows from Lemma 4.13, Lemma 4.14 and 4.15.  $\square$

Next we study the stable splitting of  $BG$  for some  $G$  related to the linear group  $L_3(p)$ . There are 6 saturated fusion systems related to  $L_3(p)$  [12, p. 46, Table 1.1].

$W_G(E)$	$ \mathcal{F}_G^{ec}\text{-rad} $	Group	$p$
$H$	$1+1$	$L_3(p)$	$3 \mid (p-1)$
$H\langle w \rangle$	2	$L_3(p) : 2$	$3 \mid (p-1)$
$T$	$1+1$	$L_3(p).3$	$3 \mid (p-1)$
$T\langle w \rangle$	2	$L_3(p).S_3$	$3 \mid (p-1)$
$T$	$1+1$	$L_3(p)$	$3 \nmid (p-1)$
$T\langle w \rangle$	2	$L_3(p) : 2$	$3 \nmid (p-1)$

We determine the stable splittings of these 6 groups. Note that the these results, with the results in [15], give a complete information on the splitting for fusion systems on  $E$  with  $|\mathcal{F}^{ec}\text{-rad}| \geq 2$  by the classification in [12].

Let

$$X = X_{0,0} \vee 2X_{p-1,0} \vee (\vee_{1 \leq i \leq (p-1)/2} X_{2i,p-1-i}) \vee M(2)$$

and

$$X' = X_{0,0} \vee X_{p-1,0} \vee (\vee_{1 \leq j \leq (p-1)/4} X_{4j,p-1-2j}) \vee M(2).$$

**Theorem 4.17.** *Suppose that  $W_G(E) = T$  and  $\mathcal{F}_G^{ec}\text{-rad} = \{A_0, A_\infty\}$ . If  $3 \nmid p-1$  then  $BG$  is stably homotopic to  $X$ . If  $p-1 = 3m$ , then  $BG$  is stably homotopic to  $X \vee L(2, m) \vee L(2, 2m)$ .*

*Proof.* By Lemma 4.4 and Lemma 4.10,  $L(1, q)$  ( $1 \leq q \leq p-2$ ) is not contained in  $BG$ . Since  $A_i$  ( $1 \leq i \leq p-1$ ) are  $T$ -conjugate, there are three conjugacy classes of maximal elementary abelian  $p$ -subgroups and two of them consist of  $\mathcal{F}_G^{ec}$ -radical subgroups. From Lemma 4.6 and Lemma 4.7, just one  $L(2, 0)$  (and one  $L(1, 0)$ ) is contained in  $BG$ . Moreover if 3 does not divide  $p-1$ , then  $L(2, q)$  is not contained in  $BG$  for each  $1 \leq q \leq p-2$  from Lemma 4.9.

Moreover, by Lemma 4.5 and Lemma 4.10,

$$n(G)_{l,k} = \begin{cases} 1 & (l = 2i, k = p-1-i, 1 \leq i \leq \frac{p-1}{2}) \\ 2 & (l = p-1, k = 0) \\ 0 & (\text{otherwise}). \end{cases}$$

On the other hand, if  $p-1 = 3m$ , then  $m(G, 2)_m = m(G, 2)_{2m} = 1$  by Proposition 4.16. This completes the proof.  $\square$

**Theorem 4.18.** *Suppose that  $W_G(E) = T\langle w \rangle$  and  $\mathcal{F}_G^{ec}\text{-rad} = \{A_0, A_\infty\}$ . If  $3 \nmid p-1$  then  $BG$  is stably homotopic to  $X'$ . If  $p-1 = 3m$ , then  $BG$  is stably homotopic to  $X' \vee L(2, m) \vee L(2, 2m)$ .*

*Proof.* The proof is similar to that of previous Theorem. By Lemma 4.4 and Lemma 4.10,  $L(1, q)$  ( $1 \leq q \leq p-2$ ) is not contained in  $BG$ . Since  $A_i$  ( $1 \leq i \leq p-1$ ) are  $T$ -conjugate, there are two conjugacy classes of maximal elementary abelian  $p$ -subgroups and one of them consists of  $\mathcal{F}_G^{ec}$ -radical subgroups. From Lemma 4.6 and Lemma 4.7, just one  $L(2, p-1)$  (and one  $L(1, p-1)$ ) is contained in  $BG$ . Moreover if 3 does not divide  $p-1$ , then  $L(2, q)$  is not contained in  $BG$  for each  $1 \leq q \leq p-2$  from Lemma 4.9.

Moreover, by Lemma 4.5 and Lemma 4.10,

$$n(G)_{l,k} = \begin{cases} 1 & (l = 4j, k = p-1-2j, 1 \leq j \leq \frac{p-1}{4}) \\ 1 & (l = p-1, k = 0) \\ 0 & (\text{otherwise}). \end{cases}$$

On the other hand, if  $p-1 = 3m$ , then  $m(G, 2)_m = m(G, 2)_{2m} = 1$  by Proposition 4.16. This completes the proof.  $\square$

Next assume that  $p-1 = 3m$ . Let  $m = 2n$ .

**Theorem 4.19.** *Suppose that  $W_G(E) = H$  and  $\mathcal{F}_G^{ec}\text{-rad} = \{A_0, A_\infty\}$ . Then  $BG$  is stably homotopic to*

$$X_{0,0} \vee 4X_{p-1,0} \vee 2(\bigvee_{n \leq i < 3n} X_{2i, 3n-i}) \vee (\bigvee_{1 \leq i < m} X_{2i, 3m-i})$$

$$\vee 3(\bigvee_{m \leq i \leq 3n} X_{2i, 3m-i}) \vee 3(M(2) \vee L(2, m) \vee L(2, 2m)).$$

*Proof.* By Lemma 4.4 and Lemma 4.11,  $m(G, 1)_q = 0$  for  $1 \leq q \leq p-2$ . On the other hand, by Lemma 4.6 and Corollary 4.7,  $m(G, 1)_0 = m(G, 2)_0 = 5-2 = 3$ . By Lemma 4.9,  $m(G, 2)_k = 0$  for  $1 \leq k \leq p-2$ ,  $k \neq m, 2m$ . By Proposition 4.16,  $m(G, 2)_m =$

$m(G, 2)_{2m} = 3$ . The multiplicity  $n(G)_{i,q}$  is obtained by Lemma 4.11. By Lemma 4.11,

$$n(G)_{i,q} = \begin{cases} 4 & (l = p - 1, q = 0) \\ 2 & (l = 2i, q = 3n - i, n \leq i < 3n) \\ 1 & (l = 2i, q = (p - 1) - i, 1 \leq i < m) \\ 3 & (l = 2i, q = (p - 1) - i, m \leq i \leq 3n) \\ 0 & (\text{otherwise}). \end{cases}$$

□

**Theorem 4.20.** *Suppose that  $W_G(E) = H\langle w \rangle$  and  $\mathcal{F}_G^{ec}\text{-rad} = \{A_0, A_\infty\}$ . Then  $BG$  is stably homotopic to*

$$X_{0,0} \vee 2X_{p-1,0} \vee (\bigvee_{n \leq i < 3n} X_{2i,3n-i}) \vee (\bigvee_{1 \leq j \leq 3n/2} X_{4j,3m-2j}) \vee (\bigvee_{m \leq i \leq 3n} X_{2i,(p-1)-i}) \\ \vee 2(M(2) \vee L(2, m) \vee L(2, 2m)).$$

*Proof.* By Lemma 4.4 and Lemma 4.12,  $m(G, 1)_q = 0$  for  $1 \leq q \leq p - 2$ . By Lemma 4.6 and Corollary 4.7,  $m(G, 1)_0 = m(G, 2)_0 = 3 - 1 = 2$ . By Lemma 4.9,  $m(G, 2)_k = 0$  for  $1 \leq k \leq p - 2$ ,  $k \neq m, 2m$ . By Proposition 4.16,  $m(G, 2)_m = m(G, 2)_{2m} = 2$ . The multiplicity  $n(G)_{i,q}$  is obtained by Lemma 4.12. By Lemma 4.12,

$$n(G)_{i,q} = \begin{cases} 2 & (l = p - 1, q = 0) \\ 1 & (l = 2i, q = 3n - i, n \leq i < 3n) \\ 1 & (l = 2i, q = (p - 1) - i, 1 \leq i < m, i: \text{even}) \\ 2 & (l = 2i, q = (p - 1) - i, m \leq i \leq 3n, i: \text{even}) \\ 1 & (l = 2i, q = (p - 1) - i, m \leq i \leq 3n, i: \text{odd}) \\ 0 & (\text{otherwise}). \end{cases}$$

Moreover, consider the 3rd, 4th and 5th cases. We have

$$(\bigvee_{1 \leq i < m, i: \text{even}} X_{2i,(p-1)-i}) \vee 2(\bigvee_{m \leq i \leq 3n, i: \text{even}} X_{2i,(p-1)-i}) \vee (\bigvee_{m \leq i \leq 3n, i: \text{odd}} X_{2i,(p-1)-i}) \\ = (\bigvee_{1 \leq i \leq 3n, i: \text{even}} X_{2i,(p-1)-i}) \vee (\bigvee_{m \leq i \leq 3n} (X_{2i,(p-1)-i})) \\ = (\bigvee_{1 \leq j \leq 3n/2} X_{4j,(p-1)-2j}) \vee (\bigvee_{m \leq i \leq 3n} X_{2i,(p-1)-i}).$$

This completes the proof. □

Next we consider the specific case, that is,  $p = 7$ . We give a result which supplements the result on splitting for  $p = 7$  in [15].

**Example 4.21.** *Let  $p = 7$ ,  $p - 1 = 6$ ,  $m = 2$ ,  $n = 1$ . Suppose that  $\mathcal{F}_G^{ec}\text{-rad} = \{A_0, A_\infty\}$ .*

(1) *If  $W_G(E) = T$ , then*

$$BG \sim X_{0,0} \vee X_{2,5} \vee X_{4,4} \vee 2X_{6,0} \vee X_{6,3} \vee M(2) \vee L(2, 2) \vee L(2, 4).$$

(2) *If  $W_G(E) = T\langle w \rangle$ , then*

$$BG \sim X_{0,0} \vee X_{4,4} \vee X_{6,0} \vee M(2) \vee L(2, 2) \vee L(2, 4).$$

(3) *If  $W_G(E) = H$ , then*

$$BG \sim X_{0,0} \vee 2X_{2,2} \vee X_{2,5} \vee 2X_{4,1} \vee 3X_{4,4} \vee 4X_{6,0} \vee 3X_{6,3} \\ \vee 3(M(2) \vee L(2, 2) \vee L(2, 4)).$$

(4) If  $W_G(E) = H\langle w \rangle$ , then

$$BG \sim X_{0,0} \vee X_{2,2} \vee X_{4,1} \vee 2X_{4,4} \vee 2X_{6,0} \vee X_{6,3}$$

$$\vee 2(M(2) \vee L(2, 2) \vee L(2, 4)).$$

Let  $G_1$  and  $G_2$  be finite groups with Sylow  $p$ -subgroup  $E$ . If  $\mathcal{F}_{G_1}$  is (isomorphic to) a subfusion system of  $\mathcal{F}_{G_2}$ , then  $BG_1 \sim BG_2 \vee X$  for some summand  $X$  of  $BG_1$ . In this case, we write

$$G_2 \xleftarrow{X} G_1$$

We use same notation for fusion systems.

In [15], the second author considered the graphs related to the splitting of sporadic simple groups and some exotic fusion system for  $p = 7$  and obtained the following.

**Theorem 4.22** ([15, Theorem 9.4]). *Let  $p = 7$ . We have the following two sequences:*

$$X_{0,0} \vee X_{4,4} \sim RV_3 \xleftarrow{X_{2,2} \vee X_{6,0}} RV_2 \xleftarrow{M(2) \vee L(2,2) \vee L(2,4)} O'N : 2 \xleftarrow{X_{4,1} \vee X_{4,4} \vee X_{6,0} \vee X_{6,3}} O'N$$

$$X_{0,0} \vee X_{4,4} \vee X_{6,0} \sim RV_1 \xleftarrow{M(2)} Fi_{24} \xleftarrow{X_{2,2} \vee X_{6,0} \vee X_{6,3}} Fi'_{24} \xleftarrow{M(2) \vee L(2,2) \vee L(2,4)} He : 2$$

$$\xleftarrow{X_{3,0} \vee X_{3,3} \vee X_{5,2} \vee X_{5,5} \vee L(1,3) \vee L(2,3)} He.$$

where  $RV_1, RV_2, RV_3$  are the exotic fusion systems of Ruiz and Viruel [12].

Now we add more information on the splittings for  $p = 7$ .

**Theorem 4.23.** *Let  $p = 7$ . We have the following diagram:*

$$\begin{array}{ccccc} RV_1 & \xleftarrow{M(2) \vee \tilde{L}} & L_3(7).S_3 & \xleftarrow{Y'} & L_3(7).3 \\ \uparrow Y \vee Z \vee M(2) \vee \tilde{L} & & \uparrow Y \vee Z \vee M(2) \vee \tilde{L} & & \uparrow 2(Y \vee Z \vee M(2) \vee \tilde{L}) \\ O'N & \xleftarrow{M(2) \vee \tilde{L}} & L_3(7) : 2 & \xleftarrow{Y \vee Y' \vee Z \vee M(2) \vee \tilde{L}} & L_3(7) \\ \downarrow Y \vee Z \vee M(2) \vee \tilde{L} & & \downarrow Y \vee Z \vee M(2) \vee 2\tilde{L} & & \downarrow Y \vee Y' \vee 2Z \vee 2M(2) \vee 3\tilde{L} \\ RV_1 & \xleftarrow{M(2)} & Fi_{24} & \xleftarrow{Y} & Fi'_{24} \end{array}$$

where

$$Y = X_{2,2} \vee X_{6,0} \vee X_{6,3}, \quad Y' = X_{2,5} \vee X_{6,0} \vee X_{6,3}, \quad Z = X_{4,1} \vee X_{4,4}$$

$$\tilde{L} = L(2, 2) \vee L(2, 4).$$

*Proof.* We have the following table by [12, Lemma 4.9, Lemma 4.16].

group (fusion system)	$\text{Out}_{\mathcal{F}}(E)$	$\mathcal{F}^{ec}\text{-rad}$	$\text{Out}_{\mathcal{F}}(A)$
$L_3(7)$	$6 \times 2 = \langle 3I, u \rangle$	$\{A_0\}\{A_\infty\}$	$\text{SL}_2(7) : 2$
$L_3(7)$	$6 \times 2 = \langle 3I, w \rangle$	$\{A_1\}\{A_6\}$	$\text{SL}_2(7) : 2$
$L_3(7) : 2$	$(6 \times 2) : 2 = \langle 3I, u, w \rangle$	$\{A_0, A_\infty\}$	$\text{SL}_2(7) : 2$
$L_3(7) : 2$	$(6 \times 2) : 2 = \langle 3I, u, w \rangle$	$\{A_1, A_6\}$	$\text{SL}_2(7) : 2$
$O'N$	$(6 \times 2) : 2 = \langle 3I, u, w \rangle$	$\{A_0, A_\infty\}\{A_1, A_6\}$	$\text{SL}_2(7) : 2$
$L_3(7).3$	$6^2 = T$	$\{A_0\}\{A_\infty\}$	$\text{GL}_2(7)$
$L_3(7).S_3$	$6^2 : 2 = T\langle w \rangle$	$\{A_0, A_\infty\}$	$\text{GL}_2(7)$
$Fi'_{24}$	$6 \times S_3 = \langle 3I, s, w \rangle$	$\{A_1, A_2, A_4\}\{A_3, A_5, A_6\}$	$\text{SL}_2(7) : 2$
$Fi_{24}$	$6^2 : 2 = T\langle w \rangle$	$\{A_1, \dots, A_6\}$	$\text{SL}_2(7) : 2$
$RV_1$	$6^2 : 2 = T\langle w \rangle$	$\{A_0, A_\infty\}\{A_1, \dots, A_6\}$	$\text{GL}_2(7), \text{SL}_2(7) : 2$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

and  $T = \{\text{diag}(\alpha, \beta) \mid \alpha, \beta \in \mathbf{F}_7^\times\}$  is the subgroup of all invertible diagonal matrices. The set  $\mathcal{F}^{ec}\text{-rad}$  is separated by conjugacy classes and  $\text{Out}_{\mathcal{F}}(A)$  is described for each representative  $A$  of conjugacy classes in  $\mathcal{F}^{ec}\text{-rad}$  if they are different. Note that if we take the generators  $a$  and  $b$  of  $E$  suitably, we can obtain the two rows in the case of  $L_3(7)$  and  $L_3(7) : 2$ . For example, consider  $G = L_3(7)$ . Let  $E$  be the group of all upper triangular matrices with diagonal entry 1. The subgroups in  $\mathcal{F}_G^{ec}\text{-rad}$  are

$$\left\{ \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid \alpha, \beta \in \mathbb{F}_p \right\}, \quad \left\{ \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \mid \alpha, \beta \in \mathbb{F}_p \right\}.$$

If we take

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

then  $\mathcal{F}_G^{ec}\text{-rad} = \{A_0, A_\infty\}$  and  $\text{Out}_{\mathcal{F}_G}(E) = \langle 3I, u \rangle = H$ . On the other hand, if we take

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

then  $\mathcal{F}_G^{ec}\text{-rad} = \{A_1, A_6\}$  and  $\text{Out}_{\mathcal{F}_G}(E) = \langle 3I, w \rangle$ .

The inclusions of fusion systems are obtained by the table above. For  $Fi'_{24} \leftarrow L_3(7)$  and  $Fi_{24} \leftarrow L_3(7) : 2$ , we use the second rows of  $L_3(7)$  and  $L_3(7) : 2$ . The information on the summands are obtained by Example 4.21 and Theorem 4.22.  $\square$

**Remark 4.24.** As we can see from Theorem 4.22 or 4.23 above,  $B(Fi_{24})$  is a stable summand of  $B(O'N)$ ,  $B(O'N) \sim B(Fi_{24}) \vee Y \vee Z \vee \tilde{L}$ , but the fusion system of  $O'N$  is not isomorphic to a subfusion system of fusion system of  $Fi_{24}$ , namely,

$$Fi_{24} \xleftarrow{Y \vee Z \vee \tilde{L}} O'N$$

does not hold.

Let  $\mathcal{F}_0 = \mathcal{F}_{O'_N}$ ,  $\mathcal{F}_1 = \mathcal{F}_{Fi_{24}}$ . By [12, Lemma 4.3], for each  $A_i \in \mathcal{F}_0^{ec}\text{-rad}$ , there exists an element of order 6 in  $\text{Out}_{\mathcal{F}_0}(E) \leq \text{GL}_2(\mathbb{F}_p)$  which has an eigenvalue 3 with eigenvector  $\begin{pmatrix} 1 \\ i \end{pmatrix}$  ( $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  if  $i = \infty$ ) and determinant 5. Hence there exists an involution in  $\text{Out}_{\mathcal{F}_0}(E)$

which has an eigenvalue  $-1$  with eigenvector  $\begin{pmatrix} 1 \\ i \end{pmatrix}$  ( $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  if  $i = \infty$ ) and determinant  $-1$ .

We may assume that  $\text{Out}_{\mathcal{F}_1}(E) = 6^2 : 2 = T\langle w \rangle$  and  $\mathcal{F}_1^{ec}\text{-rad} = \{A_1, \dots, A_6\}$  as above. Suppose that  $K = \text{Out}_{\mathcal{F}_0}(E) (\cong (6 \times 2) : 2) \leq \text{Out}_{\mathcal{F}_1}(E)$ . Then  $K$  contains exactly 4 involutions with determinant  $-1$ . Moreover  $K \supset \langle \text{diag}(-1, 1), \text{diag}(1, -1) \rangle$  since  $\langle \text{diag}(-1, 1), \text{diag}(1, -1) \rangle \triangleleft (6^2 : 2)$ . Note that  $|\mathcal{F}_0^{ec}\text{-rad}| = 4$ . Since  $\text{diag}(-1, 1)$  (resp.  $\text{diag}(1, -1)$ ) has an eigenvalue  $-1$  with eigenvector  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (resp.  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ) and determinant  $-1$ , it follows that  $A_0, A_\infty \in \mathcal{F}_0^{ec}\text{-rad}$  for any choice of  $K \leq 6^2 : 2 = T\langle w \rangle$ . Hence  $\mathcal{F}_0$  is not isomorphic to a subfusion system of  $\mathcal{F}_1$ .

## 5. SOME REMARKS ON THE CASE $p = 3$

Recall that  $H^3(E, \mathbb{Z}) = \mathbb{F}_p\{a_1, a_2\}$ . The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{j} \mathbb{F}_p \longrightarrow 0$$

induces the following short exact sequence

$$(5.1) \quad 0 \longrightarrow H^2(E, \mathbb{Z}) \xrightarrow{j_*} H^2(E, \mathbb{F}_p) \xrightarrow{\hat{\beta}} H^3(E, \mathbb{Z}) \longrightarrow 0.$$

Hence there exist elements  $a'_1, a'_2 \in H^2(E, \mathbb{F}_p)$  such that  $\hat{\beta}(a'_i) = a_i$  and

$$H^2(E, \mathbb{F}_p) = \mathbb{F}_p\{y_1, y_2, a'_1, a'_2\}.$$

We consider the action of  $\text{Out}(E) = \text{GL}_2(\mathbb{F}_p)$ . The sequence (5.1) is a sequence of  $\mathbb{F}_p \text{GL}_2(\mathbb{F}_p)$ -modules and the map  $Q_1\hat{\beta}$  induces an isomorphism of  $\mathbb{F}_p \text{GL}_2(\mathbb{F}_p)$ -modules,

$$H^2(E, \mathbb{F}_p)/H^2(E, \mathbb{Z}) \longrightarrow H^3(E, \mathbb{Z}) \longrightarrow \mathbb{F}_p\{y_1v, y_2v\} \cong S^1 \otimes \det.$$

Now, consider the sequence

$$\begin{array}{ccccc} H^{2p}(E, \mathbb{Z}) & \xrightarrow{j_*} & H^{2p}(E, \mathbb{F}_p) & \xrightarrow{\hat{\beta}} & H^{2p+1}(E, \mathbb{Z}) \\ \xrightarrow{p} & & H^{2p+1}(E, \mathbb{Z}) & \xrightarrow{j_*} & H^{2p+1}(E, \mathbb{F}_p). \end{array}$$

By taking the  $p$ -th power, we have an  $\mathbb{F}_p \text{GL}_2(\mathbb{F}_p)$ -morphism,

$$H^2(E, \mathbb{F}_p) \longrightarrow H^{2p}(E, \mathbb{F}_p).$$

Since  $\beta = j_*\hat{\beta}$  is the Bockstein homomorphism, we have  $\beta((a'_i)^p) = 0$ . Moreover, since  $j_* : H^{2p+1}(E, \mathbb{Z}) \longrightarrow H^{2p+1}(E, \mathbb{F}_p)$  is injective,  $\hat{\beta}((a'_i)^p) = 0$ . Hence, it follows that

$$(a'_i)^p \in j_*(H^{2p}(E, \mathbb{Z})) \cong H^{2p}(E).$$

On the other hand, since

$$H^{2p}(E) = CS^1 + T^1 + \mathbb{F}_p\{v\} \leftrightarrow S^1 \oplus (S^{p-2} \otimes \det) \oplus \det,$$

we see that if  $p > 3$  then  $(a'_i)^p = 0$ . Since  $H^{even}(E, \mathbb{F}_p)$  is generated by  $1, a'_1, a'_2$  as a module over  $\mathbb{F}_p \otimes H^{even}(E, \mathbb{Z})$ , it follows that

$$H^{even}(E, \mathbb{F}_p)/\sqrt{(0)} = (\mathbb{F}_p \otimes H^{even}(E, \mathbb{Z}))/\sqrt{(0)}$$

and in particular,

$$H^*(E, \mathbb{F}_p)/\sqrt{(0)} = (\mathbb{F}_p \otimes H^*(E, \mathbb{Z}))/\sqrt{(0)} = H^*(E)$$

for  $p > 3$ .

On the other hand, if  $p = 3$ , then the sequence (5.1) does not split and  $\mathbb{F}_3\{y_1, y_2\}$  is the unique nontrivial  $\mathbb{F}_3 \text{GL}_2(\mathbb{F}_3)$ -submodule of  $H^2(E, \mathbb{F}_3)$  (cf. [9, p.74]). This implies that  $(a'_i)^{p^n} \neq 0$  for any  $n > 0$  and in particular, we see that  $a'_i$  is not nilpotent.

In fact, the structure of  $H^*(E, \mathbb{F}_3)/\sqrt{(0)}$  is known by the result of Leary [9, Theorem 7] and we have the following:

**Proposition 5.1.** *Assume that  $p = 3$ . Then  $H^*(E, \mathbb{F}_3)/\sqrt{(0)}$  is generated by*

$$y_1, y_2, a'_1, a'_2, v$$

with

$$\deg y_i = \deg a'_i = 2, \deg v = 6$$

subject to the following relations:

$$y_1^3 y_2 - y_1 y_2^3 = 0$$

$$a'_1 a'_2 = a'_1 y_1 = a'_2 y_2 = y_1 y_2, (a'_1)^2 = (a'_2)^2 = a'_1 y_2 = a'_2 y_1.$$

Since  $H^4(E, \mathbb{F}_3)/(H^4(E, \mathbb{F}_3) \cap \sqrt{(0)})$  is spanned by  $y_1^2, y_1 y_2, y_2^2, (a'_1)^2$  and  $\dim_{\mathbb{F}_3} H^4(E) = 4$ ,

$$H^4(E, \mathbb{F}_3)/(H^4(E, \mathbb{F}_3) \cap \sqrt{(0)}) = H^4(E).$$

In particular,  $(a'_i)^2 \in H^4(E)$  and hence we have

$$H^*(E, \mathbb{F}_3)/\sqrt{(0)} = H^*(E) \oplus \mathbb{F}_3[v]\{\mathbb{F}_3 a'_1 + \mathbb{F}_3 a'_2\}.$$

Since  $H^2(E, \mathbb{F}_3)/H^2(E) \cong S^1 \otimes \det$  as  $\mathbb{F}_3 \text{GL}_2(\mathbb{F}_3)$ -modules,

$$(H^*(E, \mathbb{F}_3)/\sqrt{(0)})/H^*(E) \cong \mathbb{F}_3[v] \otimes (S^1 \otimes \det)$$

as  $\mathbb{F}_3 \text{GL}_2(\mathbb{F}_3)$ -modules. If  $Q$  is a proper subgroup of  $E$ , then  $H^*(Q, \mathbb{F}_3)/\sqrt{(0)} = H^*(Q)$ . Hence

$$\begin{aligned} (H^*(E, \mathbb{F}_3)/\sqrt{(0)})A_3(Q, E)A_3(E, Q) &\subset (H^*(Q, \mathbb{F}_3)/\sqrt{(0)})A_3(E, Q) \\ &= H^*(Q)A_3(E, Q) \subset H^*(E). \end{aligned}$$

In particular,  $(H^*(E, \mathbb{F}_3)/\sqrt{(0)})/H^*(E)$  is annihilated by  $A_3(Q, E)A_3(E, Q)$  for any  $Q < E$ . Hence, every composition factors of  $(H^*(E, \mathbb{F}_3)/\sqrt{(0)})/H^*(E)$  as an  $A_3(E, E)$ -module is isomorphic to  $S(E, E, S^i \otimes \det^q)$  for some  $i, q$  and we have the following:

**Proposition 5.2.**

$$(H^n(E, \mathbb{F}_3)/\sqrt{(0)})/H^n(E) \cong \begin{cases} S(E, E, S^1 \otimes \det) & (n \equiv 2 \pmod{12}) \\ S(E, E, S^1) & (n \equiv 8 \pmod{12}) \\ 0 & (\text{otherwise}). \end{cases}$$

**Corollary 5.3.** *Let  $X_{0,0}$  be the summand which corresponding to the simple module  $S(E, E, \mathbb{F}_p)$  and  $e$  be the corresponding idempotent in  $A_p(E, E)$ . Then*

$$(H^*(E, \mathbb{F}_3)/\sqrt{(0)})e \cong H^*(E)e \cong \mathbb{D}\mathbb{A}^+.$$

At last of this paper, we see more closely the cohomology  $H^*(X)$  of a summand  $X$  in the stable splitting of  $BG$  with  $E \in \text{Syl}_3(G)$  in the case  $p = 3$ . The lowest degree and some of the second lowest degree  $* > 0$  with  $H^{2*}(X) \neq 0$  are given as follows:

$$\begin{array}{ll} L(1, 1) & : |S^1| = 1 \\ L(2, 1) & : |CS^1v| = 6 \\ X_{0,0} & : |V| = 6 \\ X_{1,0} & : |S^1V| = 7, \\ X_{2,0} & : |S^2V| = 8 \end{array} \quad \begin{array}{ll} L(1, 0) & : |y^2| = 2, \\ L(2, 0) & : |S^2D_2| = 10, \\ X_{0,1} & : |v| = 3 (|Cv| = 5) \\ X_{1,1} & : |T^1| = 3 (|S^1v| = 4) \\ X_{2,1} & : |S^2v| = 5 \end{array}$$

where  $|x| = \frac{1}{2} \deg x$  for an element or a subspace of  $H^*(E)$ . First note that  $BG$  always contains  $X_{0,0}$ . The lowest degree of nonzero elements in  $H^{2*}(L(i, j))$  or  $H^{2*}(X_{i,q})$ ,  $(i, q) \neq (0, 0)$  are all different except for  $X_{0,1}$  and  $X_{1,1}$ . On the other hand we see  $H^4(X_{0,1}) = 0$  but  $H^4(X_{1,1}) \cong \mathbb{F}_3$ . Moreover  $L(1, 0)$  and  $L(2, 0)$  have same multiplicity by Lemma 4.6 and 4.7. Hence we can count the numbers of

$$L(1, 1), M(2) = L(1, 0) \vee L(2, 0), X_{1,1}, X_{0,1}, X_{2,1}, L(2, 1), X_{1,0}, X_{2,0}$$

from  $H^{2*}(G)$  for  $* = 1, 2, 4, 3, 5, 6, 7, 8$ . Thus we have the following result which is similar to Corollary 4.2 (1) (See Remark 4.8).

**Theorem 5.4.** *Let  $G_1$  and  $G_2$  be finite groups with same Sylow 3-subgroup  $E$ . If*

$$\dim H^{2n}(G_1) = \dim H^{2n}(G_2)$$

*for  $n \leq 8$ , then  $BG_1 \sim BG_2$ .*

For example, let  $G_1 = {}^2F_4(2)'$  and  $G_2 = J_4$ . Then by [15, Theorem 6.2],

$$B({}^2F_4(2)') \sim BJ_4 \vee X_{2,0}.$$

Hence  $H^{2n}({}^2F_4(2)') \cong H^{2n}(J_4)$  for  $n < 8$  and  $\dim H^{16}({}^2F_4(2)') > \dim H^{16}(J_4)$ . See [15, section 6] for details.

**Remark 5.5.** If  $G$  has a Sylow 3-subgroup  $E$ , then  $BG$  is homotopic to the classifying space of one of the groups listed in [15, Theorem 6.2]. Moreover the cohomology of each dominant summand  $X_{i,j}$  of  $BE$ , except for  $X_{1,0}$  and  $X_{1,1}$ , is deduced from the cohomology of those finite groups.

On the other hand, as we can see from the graph in [15, Theorem 6.2],  $X_{1,0}$  and  $X_{1,1}$  always appear as  $X_{1,0} \vee X_{1,1}$ . We shall give an brief explanation of this fact. Let

$$H = W_G(E) = N_G(E)/EC_G(E) \leq \text{Out}(E) = \text{GL}_2(\mathbb{F}_3).$$

Note that  $H$  is a  $3'$ -group, in fact, 2-group. The multiplicity of  $X_{1,j}$  in the stable splitting of  $BG$  is equal to  $\dim(S^1 \otimes \det^j)^H$ . We have to show that

$$\dim(S^1)^H = \dim(S^1 \otimes \det)^H.$$

We may assume that  $H \neq 1$ . Then  $\dim(S^1)^H = 1$  if and only if  $H$  is conjugate to the subgroup  $\langle \text{diag}(1, -1) \rangle$  in  $\text{GL}_2(\mathbb{F}_3)$ . Similarly  $\dim(S^1 \otimes \det)^H = 1$  if and only if  $H$  is conjugate to the subgroup  $\langle \text{diag}(1, -1) \rangle$  in  $\text{GL}_2(\mathbb{F}_3)$ . Hence we have

$$\dim(S^1)^H = \dim(S^1 \otimes \det)^H$$

and this implies that  $X_{1,0}$  and  $X_{1,1}$  appear in  $BG$  with same multiplicity.



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